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Integral Constraints Pursuit Problem for 3-System of Differential Equations in Hilbert Space

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Abstract: This paper investigates a pursuit game of an infinite 3-system of differential equations involving one pursuer and one evader. Integral constraints are imposed on the players' control parameters where the resource of the pursuer is greater than that of the evader. The aim of the pursuer is to bring the initial state ξ^0 of the system into another non-zero state ξ^1 of the system at some time on a finite time interval to complete the pursuit. We constructed a control function needed in solving control problems of the system and developed an admissible pursuer's strategy to solve the pursuit problem. Pursuit is shown to be completed in the game as the state of the system reaches state ξ^1 at a finite time $\theta > 0$.

Keywords: 3-System, Control, Infinite, Integral Constraints, Strategy

1 Introduction

The growing necessity to resolve conflicts between two competing parties with contradictory goals led to the development of differential game theory by Isaacs [1] in 1965, where the conflicts are modelled by either partial differential equations or ordinary differential equations. Its significant contribution extends across a wide range of fields, including engineering, economics, military strategy, and transportation.

Various differential game problems described by some ordinary differential equations, were studied in a finite dimensional space such as $\mathbb{R}^2$ or a more general Euclidean space $\mathbb{R}^n, n > 2$ as in the works of [2-5]. However, a mathematical model described by a system of partial differential equations can be reduced to a model described by an infinite system of ordinary differential equations by using the method of decomposition [6-11]. Therefore, such model of differential game must be studied in Hilbert space which is an infinite dimensional space. In recent years, many differential games, described by an infinite system of ordinary differential equations, were investigated on its own (for example in [12-22]), including this study.

In numerous studies of pursuit differential games, the pursuit is said to be completed when the initial position of the system reaches the origin of the space, as can be found in the works [12-18]. For example, the work of Ibragimov et al. [15] focused on solving a pursuit differential game described by an infinite 1-system of differential equations as follows:

$$\dot{z_k} = -\lambda_k z_k + u_k - v_k, \quad z_k(0) = z_{k0} \tag{1}$$

where $z_k, u_k, v_k \in \mathbb{R}$ and $\lambda_k \in \mathbb{R}^+$ for $k = 1,2, \dots$. The players' control functions are subjected to geometric constraints. The pursuit is said to be completed if the geometric position of the state of the system coincides with the origin of l_2 space at some time. An admissible pursuer's strategy and a formula of guaranteed pursuit time of the game were established.

Ibragimov et al. [16] continued to study a differential game (1), but each coefficient $\lambda_k, k = 1,2, \dots$, now complies to condition $\lambda_1 \leq \lambda_2 \leq \cdots$. A strategy of satisfying geometric constraints was constructed for the pursuer and an equation of guaranteed pursuit time was found.

The study continues to a higher-level system of differential equations with the work of Tukhtasinov et al. [17] which examined a pursuit game in a higher-level system of differential equations that is an infinite 2-system of differential equations defined by

$$\begin{aligned} \dot{x_k} &= -\alpha_k x_k - \beta_k y_k - u_{k1} + v_{k1}, \quad x_k(0) = x_{k0}, \\ \dot{y_k} &= \beta_k x_k - \alpha_k y_k - u_{k2} + v_{k2}, \quad y_k(0) = y_{k0}, \end{aligned} \tag{2}$$

where $\alpha_k, \beta_k \in \mathbb{R}, \alpha_k \geq 0$ for $k = 1,2, \dots$. Geometric constraints restricted the players' control functions where the resource ρ of the pursuer is greater than σ that of the evader. Pursuer's strategy was developed, and the pursuit was successfully completed. An estimate of a guaranteed pursuit time together with a guaranteed evasion time of the game were also obtained.

Ibragimov et al. [18], on the other hand, investigated a pursuit game of 2-system with the system given in the form of

$$\begin{aligned} \dot{x_k} &= \gamma_k x_k - \delta_k y_k + u_{k1} - v_{k1}, \quad x_k(0) = x_{k0}, \\ \dot{y_k} &= \delta_k x_k + \gamma_k y_k + u_{k2} - v_{k2}, \quad y_k(0) = y_{k0}, \end{aligned} \tag{3}$$

where $\gamma_k \geq 0, \delta_k \in \mathbb{R}$. The control functions of pursuer and evader were constrained by integral constraints. In this work, the pursuer seeks to shift the state of the system into the origin as the evader attempts to avoid this. Optimal strategies were established for the players and thus, optimal pursuit time of the game was obtained.

In addition to all of these investigations, there were also works that examined pursuit differential
games where the completion of pursuit is achieved when the initial state of the system is shifted into another non-zero state. For example, the work of Waziri et al. [19] investigated a pursuit game described by an infinite system of the first order,

$$\dot{z_k}(t) + \mu_k z_k(t) = w_k(t), \ z_k(0) = z_k^0 \tag{4}$$

where $z_k, w_k \in \mathbb{R}, k = 1, 2, \dots,$ $z^0 = (z_1^0, z_2^0, \dots) \in l_2$ and w_k is the control parameter of the system. The set of pairs (z^0, z^1) involving initial state $z^0 = (z_1^0, z_2^0, \dots) \in l_2$ and a non-zero state $z^1 = (z_1^1, z_2^1, \dots) \in l_2$ of the system were established and utilised to construct the sufficient condition for the game. Both geometric constraints and integral constraints were imposed on the players' control functions but were studied separately. The solution to the control problem and the construction of the pursuer's strategy was obtained.

Furthermore, the system in [19] was described as coordinate-wise in the study of Waziri and Ibragimov [20], where the respective control function of the pursuer and evader are subjected coordinate-wise to integral constraints.

$$\sum_{k=1}^{\infty} \int_0^{\theta} |u_k(t)|^2 \, dt \leq \rho_k, \sum_{k=1}^{\infty} \int_0^{\theta} |v_k(t)|^2 \, dt \leq \sigma_k, \tag{5}$$

with $\rho_k,\ \sigma_k > 0$ and $\rho_k > \sigma_k$ for all $k = 1,2,\ldots$. A strategy for the pursuer in bringing z^0 into z^1 at time $\theta > 0$ was developed.

The problem of differential game described by an infinite two-system of differential equations continued to be solved as can be seen in Ibragimov et al. [21]. In the work, the game was described as follows:

$$\dot{x}_k = -\alpha_k x_k - \beta_k y_k + u_{k1} - v_{k1},\ \ x_k(0) = x_{k0},$$
$$\dot{y}_k = \beta_k x_k - \alpha_k y_k + u_{k2} - v_{k2},\ \ y_k(0) = y_{k0}, \tag{6}$$

where $\alpha_k \geq 0,\ \beta_k \in \mathbb{R}$ for $k = 1,2,\ldots$. Integral constraints was imposed on the control functions of both pursuer and evader. A similar work was carried out in Waziri et al. [22] for the game defined by (6) but with geometric constraints restricting the players' control functions. In both works, it was shown that the pursuit can be completed by constructing feasible strategy for the pursuer.

The reviews above show some studies that were conducted to solve pursuit differential games described by infinite 1 and 2-system of differential equations, which entail advancing the state of the system either into the origin or into another non-zero state of Hilbert space which has been the subject of numerous investigations. However, a pursuit game could occur in a more complex system, that is of a higher level of system of differential equations. The method of solving pursuit game of any level of infinite system cannot be generalised as every system has its unique fundamental matrix and must be dealt on its own.

We are, therefore, motivated to study a more complex problem of solving the pursuit game which is defined by a 3-system of differential equations in the present research. The pursuer's goal is to terminate the pursuit by transferring the initial state into a non-zero state at some time. The game between one pursuer and one evader is set to occur in Hilbert space l_2 which satisfies:

$$< \zeta, \eta > = \sum_{k=1}^{\infty} \zeta_k \eta_k < \infty,$$
$$||\zeta|| = < \zeta, \zeta > = \zeta_k^2 < \infty. \tag{7}$$

The equation of the game is given by

$$\dot{x}_k = -\mu_k x_k - u_{k1} + v_{k1},\ x_k(0) = x_k^0,$$
$$\dot{y}_k = -\lambda_k y_k - \delta_k z_k - u_{k2} + v_{k2},\ y_k(0) = y_k^0, \tag{8}$$
$$\dot{z}_k = \delta_k y_k - \lambda_k z_k - u_{k3} + v_{k3},\ z_k(0) = z_k^0,$$

where $\mu_k,\ \lambda_k \geq 0,\ \delta_k \in \mathbb{R},\ u_{kj}, v_{kj} \in \mathbb{R}$ for $k = 1,2,\ldots,$ and $j = 1,2,3,$ with control parameter $u = (u_1,\ u_2,\ \ldots) = (u_{11},\ u_{12},\ u_{13},\ u_{21}, u_{22},\ u_{23}, \ldots) \in l_2$ of pursuer and control parameter $v = (v_1, v_2, \ldots) = (v_{11},\ v_{12},\ v_{13},\ v_{21}, v_{22}, v_{23}, \ldots) \in l_2$ of evader.

2 Some Preliminaries

We examined a pursuit differential game defined by infinite 3-system of differential equations (8). Let

$$x^0 = (x_1^0, x_2^0, x_3^0, \ldots) \in l_2, \quad y^0 = (y_1^0, y_2^0, y_3^0, \ldots) \in l_2, \quad z^0 = (z_1^0, z_2^0, z_3^0, \ldots) \in l_2,$$

be the initial state of the system and

$$x^1 = (x_1^1, x_2^1, x_3^1, \ldots) \in l_2, \quad y^1 = (y_1^1, y_2^1, y_3^1, \ldots) \in l_2, \quad z^1 = (z_1^1, z_2^1, z_3^1, \ldots) \in l_2,$$

be another non-zero state of the system.

The game is said to be completed if the state ξ^0 reaches ξ^1 at some finite time. We present the sufficient conditions to solve the control problem and ensure the completion of pursuit to take place. Note that the state ξ^1 considered here does not equal to the origin of space l_2. The following definitions were derived from the work of Ibragimov et al. [21].

Definition 2.1. The admissible control $w(\cdot) = (w_1(\cdot), w_2(\cdot), \dots) \in l_2, w : [0, \theta] \to l_2$ satisfies:

$$\sqrt{\sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} \left|w_{kj}(t)\right|^2 dt} \leq \rho_0,$$

for a given positive ρ_0 where $w_k = (w_{k1}, w_{k2}, w_{k3})$ for $k = 1,2, \dots$.

One can note that the set of all admissible controls, set of admissible pursuer's control and set of admissible evader's control is denoted by $S(\rho_0), S(\rho)$ and $S(\sigma)$, respectively.

Definition 2.2. The admissible pursuer's control $u(\cdot)$ and admissible evader's control $v(\cdot)$ satisfy:

$$\sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} \left|u_{kj}(t)\right|^2 dt \leq \rho^2, \quad \sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} \left|v_{kj}(t)\right|^2 dt \leq \sigma^2$$

for a given positive ρ and σ where $\rho > \sigma$.

Definition 2.3. The admissible pursuer's strategy $U(\cdot, v) = (U_1(\cdot, v), U_2(\cdot, v), \dots), U : [0, \theta] \times l_2 \to l_2$ satisfies**:**

$$\sqrt{\sum_{k=1}^{\infty} \int_0^{\theta} \sum_{j=1}^{3} |U_k(t, v_k(t))|^2 dt} \leq \rho$$

for a given positive ρ such that $U_k(t, v_k(t)) = v_k(t) - w_k(t), U_{kj}, v_{kj}, w_{kj}, k = 1, 2, \dots$, and $j = 1, 2, 3$, where $w(\cdot) \in S(\rho - \sigma), v(\cdot) \in S(\sigma)$.

Definition 2.4. Pursuit is said to be completed in game (8) if, for any admissible control of evader $v(\cdot) \in S(\sigma)$, there exists an admissible strategy of pursuer $U(\cdot, v(\cdot))$ such that $\xi(t) = \xi^1$ for some time $t, t \in [0, \theta]$.

We know that the system of differential equations (8) has a unique solution $\xi(\cdot) = (\xi_1(\cdot), \xi_2(\cdot), \dots) \in l_2$ given by

$$\xi_k(t) = M_k(t)\xi_k^0 + \int_0^t M_k(t-s)\big(-u_k(s) + v_k(s)\big)\, ds, \tag{9}$$

Where:

$$\begin{pmatrix} e^{-\mu_k t} & 0 & 0 \\ 0 & e^{-\lambda_k t}\cos\delta_k t & -e^{-\lambda_k t}\sin\delta_k t \\ 0 & e^{-\lambda_k t}\sin\delta_k t & e^{-\lambda_k t}\cos\delta_k t \end{pmatrix} \tag{10}$$

for $k = 1, 2, \dots$.

It is easy to show that the matrix $M_k(t), k = 1,2,\dots$, obeys the following: (i) $M_k(t-s) = M_k(t)M_k(-s) = M_k(-s)M_k(t)$, (ii) $M_k^{-1}(t) = M_k(-t)$, (iii) $|M_k(t)\xi_k| \leq e^{-\alpha_k t}\,|\xi_k|$ where $\alpha_k = min\{\mu_k, \lambda_k\}$.

We denote that:

$$\xi(t) = (\xi_1(t), \xi_2(t), \dots) = (x_1(t), y_1(t), z_1(t), x_2(t), y_2(t), z_2(t), \dots),$$

$$|\xi(t)|| = \sqrt{\sum_{k=1}^{\infty}(x_k^2(t) + y_k^2(t) + z_k^2(t))}$$

$$\xi_k(t) = \big(x_k(t), y_k(t), z_k(t)\big), \quad |\xi_k(t)| = \sqrt{x_k^2(t) + y_k^2(t) + z_k^2(t)},$$

$$\xi^0 = (\xi_1^0, \xi_2^0, \dots) = (x_1^0, y_1^0, z_1^0, x_2^0, y_2^0, z_2^0, \dots),$$

$$||\xi^0|| = \sqrt{\sum_{k=1}^{\infty}((x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2)},$$

(11)

$$\xi_k^0 = \big(x_k^0, y_k^0, z_k^0\big), \quad |\xi_k^0| = \sqrt{(x_k^0)^2 + (y_k^0)^2 + (z_k^0)^2},$$

$$\xi^1 = (\xi_1^1, \xi_2^1, \dots) = (x_1^1, y_1^1, z_1^1, x_2^1, y_2^1, z_2^1, \dots),$$

$$||\xi^1|| = \sqrt{\sum_{k=1}^{\infty}((x_k^1)^2 + (y_k^1)^2 + (z_k^1)^2)},$$

$$\xi_k^1 = \big(x_k^1, y_k^1, z_k^1\big), \quad |\xi_k^1| = \sqrt{(x_k^1)^2 + (y_k^1)^2 + (z_k^1)^2}.$$

The existence of a unique solution $\xi(\cdot)$ for system (8) was established clearly in Madhavan et al. [23]. The problem of this research is to find solutions for control problem and pursuit problem described by 3-system of differential equations (8).

3 Results

A Control Problem of the System

We first solve control problem of the system. For this, we developed a control $w(\cdot)$ which is then shown to be admissible so that it can be used to drag initial state ξ^0 into another state ξ^1 at finite time θ. We express the solution of the system as:

$$\xi_k(t) = M_k(t)\,\xi_k^0 + \int_0^t M_k(t-s) w_k(s) ds. \tag{12}$$

Consider the equation, for each $k = 1, 2, \dots$,

$$P_k(t) = \int_0^t M_k(-s)M_k^*(-s)ds = \begin{pmatrix} \int_0^t e^{2\mu_k s}\,ds & 0 & 0 \\ 0 & \int_0^t e^{2\lambda_k s}\,ds & 0 \\ 0 & 0 & \int_0^t e^{2\lambda_k s}\,ds \end{pmatrix} \quad (13)$$

and

$$P_k^{-1}(t) = \begin{pmatrix} \left(\int_0^t e^{2\mu_k s}\,ds\right)^{-1} & 0 & 0 \\ 0 & \left(\int_0^t e^{2\lambda_k s}\,ds\right)^{-1} & 0 \\ 0 & 0 & \left(\int_0^t e^{2\lambda_k s}\,ds\right)^{-1} \end{pmatrix}. \quad (14)$$

Also,

$$Q_k(t) = \begin{pmatrix} e^{2\mu_k s} & 0 & 0 \\ 0 & e^{2\lambda_k s} & 0 \\ 0 & 0 & e^{2\lambda_k s} \end{pmatrix}. \quad (15)$$

Let

$$\sum_{k=1}^{\infty} 2\left(\xi_k^0\right)^* P_k^{-1}(\theta)\xi_k^0 + 2\left(\xi_k^1\right)^* Q_k(\theta)P_k^{-1}(\theta)\xi_k^1 = \rho_0^2 \quad (16)$$

where $\theta > 0$.

Theorem 3.1. Let (16) be fulfilled. Then, there exists an admissible control function $w(\cdot) \in S(\rho_0)$ to transfer the state of system (8) into another state ξ^1 at finite time θ.

Proof. We divided the proof of the theorem into three parts.

I) Construction of control

We define the control as:

$$w_k(t) = \begin{cases} -M_k^*(-t)(P_k^{-1}(\theta)\xi_k^0 - M_k(-\theta)P_k^{-1}(\theta)\xi_k^1), & t \in [0,\theta], \\ 0, & t \in (\theta,\infty), \end{cases} \quad (17)$$

for each k.

II) Admissibility of constructed control (17)

Show that control function $w(\cdot)$ is admissible. In our study, control $w(\cdot)$ is considered admissible if it satisfies integral constraints. We used the inequality $|a+b|^2 \le 2|a|^2 + 2|b|^2$ and obtained:

$$\sum_{k=1}^{\infty} \int_0^{\theta} |w_k(s)|^2\,ds$$

$$= \sum_{k=1}^{\infty} \int_0^{\theta} \left|-M_k^*(-s)(P_k^{-1}(\theta)\xi_k^0 - M_k(-\theta)P_k^{-1}(\theta)\xi_k^1)\right|^2 ds$$

$$\leq\ 2\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\right|^2 \text{ ds} + 2\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\right|^2 \text{ ds}. \qquad (18)$$

For $\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\right|^2$ ds, we did obtain that:

$$\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\right|^2 \text{ ds}$$
$$= \sum_{k=1}^{\infty}\int_0^{\theta}\left(\int_0^{\theta}\left(x_k^0\right)^2 e^{2\mu_k s}\left(\int_0^{\theta} e^{2\mu_k s}\, ds\right)^{-2} + \left(y_k^0\right)^2 e^{2\lambda_k s}\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-2} + \left(z_k^0\right)^2 e^{2\lambda_k s}\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-2}\right)$$

$$= \sum_{k=1}^{\infty}\int_0^{\theta}\left(\left(x_k^0\right)^2\left(\int_0^{\theta} e^{2\mu_k s}\, ds\right)^{-1} + \left(y_k^0\right)^2\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-1} + \left(z_k^0\right)^2\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-1}\right)$$

$$= \sum_{k=1}^{\infty}\left(\xi_k^0\right)^* P_k^{-1}(\theta)\xi_k^0.$$

Whereas, for $\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\right|^2$ ds, we carried out the following,

$$\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k(-\theta)M_k^*(-s)P_k^{-1}(\theta)\xi_k^1\right|^2 ds$$

$$= \sum_{k=1}^{\infty}\left(\left(x_k^1\right)^2\left(\int_0^{\theta} e^{2\mu_k s}\, ds\right)^{-2}\int_0^{\theta} e^{2\mu_k(\theta+s)}\, ds + \left(y_k^1\right)^2\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-2}\int_0^{\theta} e^{2\lambda_k(\theta+s)}\, ds + \left(z_k^1\right)^2\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-2}\int_0^{\theta} e^{2\lambda_k(\theta+s)}\, ds\right)$$
$$= \sum_{k=1}^{\infty}\left(\left(x_k^1\right)^2 e^{2\mu_k\theta}\left(\int_0^{\theta} e^{2\mu_k s}\, ds\right)^{-1} + \left(y_k^1\right)^2 e^{2\lambda_k\theta}\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-1} + \left(z_k^1\right)^2 e^{2\lambda_k\theta}\left(\int_0^{\theta} e^{2\lambda_k s}\, ds\right)^{-1}\right)$$
$$= \left(\xi_k^1\right)^* Q_k(\theta)P_k^{-1}(\theta)\xi_k^1.$$

By substituting information back into (18), we get that:

$$\sum_{k=1}^{\infty}\int_0^{\theta}|w_k(s)|^2\, ds \leq 2\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\right|^2 ds + 2\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\right|^2 ds$$

$$= 2\sum_{k=1}^{\infty}\left(\xi_k^0\right)^* P_k^{-1}(\theta)\xi_k^0 + 2\sum_{k=1}^{\infty}\left(\xi_k^1\right)^* Q_k(\theta)P_k^{-1}(\theta)\xi_k^1 = \rho_0^2\,.$$

Thus, $w(\cdot) \in S(\rho_0)$.

III) The transferring of state into another non-zero state

We then show that the state of system (8) is transferred from ξ^0 to ξ^1 by using (17) into (12) that is

$$\begin{aligned}\xi_k(\theta) = {} & M_k(\theta)\ \Big(\xi_k^0 - \int_0^\theta M_k(-s)M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\, ds + \\ & \int_0^\theta M_k(-s)M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\, ds \ + \int_0^\theta M_k(-s)v_k(s)\, ds - \int_0^\theta M_k(-s)v_k(s)\, ds \Big) \\ = {} & M_k(\theta)\ \left(\xi_k^0 - \int_0^\theta M_k(-s)M_k^*(-s)\, ds\, P_k^{-1}(\theta)\xi_k^0 + \int_0^\theta M_k(-s)M_k^*(-s)P_k^{-1}(\theta)\xi_k^1\, ds\right) \\ = {} & M_k(\theta)\ \left(\xi_k^0 - P_k(\theta)P_k^{-1}(\theta)\xi_k^0 + \int_0^\theta M_k(-s)M_k^*(-s)\, ds\, P_k^{-1}(\theta)\xi_k^1\right) \\ = {} & M_k(\theta)\ \left(\xi_k^0 - \xi_k^0 + P_k(\theta)P_k^{-1}(\theta)\xi_k^1\right) \\ = {} & M_k(\theta)\ \left(M_k^{-1}(\theta)\xi_k^1\right) = \xi_k^1.\end{aligned}$$

This finishes the proof the theorem.

B Pursuit Differential Game

We now extend to solve pursuit problem defined by system (8) which involves pursuers and evaders. By applying control obtained in control problem, we established admissible strategy required by the pursuers in bringing the initial state ξ^0 into another state ξ^1 at finite time θ.

Here, we have

$$\sum_{k=1}^{\infty} 2\left(\xi_k^0\right)^* P_k^{-1}(\theta)\xi_k^0 + 2\left(\xi_k^1\right)^* Q_k(\theta)P_k^{-1}(\theta)\xi_k^1 = (\rho-\sigma)^2. \qquad (19)$$

In this section, the state of the game includes the pursuer's strategy that is

$$\xi_k(t) = M_k(t)\,\xi_k^0 + \int_0^t M_k(t-s)\left(-U_k\big(s, v_k(s)\big) + v_k(s)\right) ds. \qquad (20)$$

Theorem 3.2. Let $\rho > \sigma$ and (19) be fulfilled. Then, pursuit game (8) can be completed at finite time θ.

Proof. This theorem requires the construction of strategy for the pursuer. The proof is shown in three parts as follows.

I) Construction of pursuer's strategy

We constructed the pursuer's strategy as follows:

$$U_k(t, v_k(t)) = \begin{cases} M_k^*(-t)\left(P_k^{-1}(\theta)\xi_k^0 - M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\right) + v_k(t), & t \in [0,\theta], \\ 0, & t \in (\theta, \infty). \end{cases} \qquad (21)$$

II) Admissibility of pursuer's strategy (21)

We show that strategy (21) satisfies integral constraints where $v(\cdot) \in S(\sigma)$. We employed Minowskii's inequality and obtained:

$$\sqrt{\sum_{k=1}^{\infty}\int_0^{\theta}\left|U_k\big(s,v_k(s)\big)\right|^2 ds}$$
$$\le \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta}\left|M_k^*(-s)(P_k^{-1}(\theta)\xi_k^0 - M_k(\theta)P_k^{-1}(\theta)\xi_k^1)\right|^2 ds} + \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta}|v_k(s)|^2 ds}$$
$$\le \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta}2\left|M_k^*(-s)P_k^{-1}(\theta)\xi_k^0\right|^2 + 2\left|M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\right|^2 ds} + \sigma$$
$$= \sqrt{\sum_{k=1}^{\infty}\int_0^{\theta}2(\xi_k^0)^* P_k^{-1}(\theta)\xi_k^0 + 2(\xi_k^1)^* Q_k(\theta)P_k^{-1}(\theta)\xi_k^1} + \sigma$$
$$= \sqrt{(\rho-\sigma)^2} + \sigma = \rho.$$

Hence, pursuer's strategy (21) is admissible.

III) Completion of pursuit

We proved that pursuit can be completed by utilising strategies (21) that is

$$\begin{aligned}
\xi_k(\theta) = {} & M_k(\theta)\xi_k^0 + \int_0^{\theta}(M_k(-s)\big(-M_k^*(-s)P_k^{-1}(\theta)\xi_k^0 - M_k^*(-s)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\big) + v_k(s))\, ds \\
& + \int_0^{\theta} M_k(-s)v_k(s)\, ds \\
= {} & M_k(\theta)\Bigg(\xi_k^0 - \int_0^{\theta} M_k(-s)M_k^*(-s)ds P_k^{-1}(\theta)\,\xi_k^0\, ds \\
& + \int_0^{\theta} M_k(-s)M_k^*(-s)ds M_k(-\theta)P_k^{-1}(\theta)\,\xi_k^1\, ds\Bigg) \\
= {} & M_k(\theta)\big(\xi_k^0 - P_k(\theta)P_k^{-1}(\theta)\xi_k^0 + P_k(\theta)M_k(-\theta)P_k^{-1}(\theta)\xi_k^1\big) \\
= {} & M_k(\theta)\big(M_k(-\theta)\xi_k^1\big) = \xi_k^1.
\end{aligned}$$

Consequently, strategy (21) guarantees the pursuit to be completed in the game at finite time θ. The proof ends.

4 Conclusion

This study examined a pursuit differential game described by an infinite first order 3-system of differential equations with integral constraints constraining the players' control functions in l_2 space. Sufficient conditions comprising of states ξ^0 and ξ^1 that are needed to steer the state of the system into another non-zero state and complete the pursuit at some finite times are constructed. The pursuer was provided with an admissible strategy and guaranteed pursuit time of the game was achieved.

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