

## **The Convex Polytopes and Homogeneous Coordinate Rings of Bivariate Polynomials**

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# **ABSTRACT**

*Concepts from algebraic geometry such as cones and fans are related to toric varieties and can be applied to determine the convex polytopes and homogeneous coordinate rings of multivariate polynomial systems. The homogeneous coordinates of a system in its projective vector space can be associated with the entries of the resultant matrix of the system under consideration. This paper presents some conditions for the homogeneous coordinates of a certain system of bivariate polynomials through the construction and implementation of the Sylvester-Bèzout hybrid resultant matrix formulation. This basis of the implementation of the Bèzout block applies a combinatorial approach on a set of linear inequalities, named 5-rule. The inequalities involved the set of exponent vectors of the monomials of the system and the entries of the matrix are determined from the coefficients of facets variable known as brackets. The approach can determine the homogeneous coordinates of the given system and the entries of the Bèzout block. Conditions for determining the homogeneous coordinates are also given and proven.* 

**Keywords:** *algebraic geometry, Bèzout resultant matrix, combinatorial, facet variable, homogeneous coordinates* 



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### **Introduction**

The fundamental objects of study in algebraic geometry are algebraic varieties, which are geometric manifestations of solutions of systems of polynomial equations. The studied classes of algebraic varieties were in plane algebraic curves, which include lines and circles, and toric varieties such as divisors and fans (collection of cones) as in Puenta [1] and Karzhemanov [2, 3]. A good reference in the field of algebraic geometry can be referred in Cox *et. al* [4, 5]. Basically, the subject of algebraic geometry is the study of systems of polynomial equations in several variables, that has a wide range of applications in science and engineering, for instance in robotic, camera motion, computer aided design (CAD) and computer aided manufacturing (CAM) systems [6]. Those systems were modeled as sparse polynomials which can be utilised by considering the combinatorial structure of the polynomials represented as the Newton polytopes, refer Cox *et. al*  [7], which is the convex set of the exponent vectors of the polynomials' variables. Recently, Dickenstein *et. al* [8] also worked on the combinatorics of polytopes but in 4-dimensional.

Ahmad and Aris [9] presented an algorithm that has applied the concepts from algebraic geometry to determine the homogeneous coordinate ring of bivariate polynomials and to compute the entries of the Bèzout matrix of the system. The algorithm terminates successfully with the correct matrix dimension and entries in comparison with the work given in Khetan [10]. In this paper, we study the effects of translating the vertices of the polytopes of the polynomial system studied in Khetan. The effects of scaling the dimension of the Newton polytope are also investigated. Thus, sufficient conditions in deriving the homogeneous coordinates of such systems will be established. be established. Thus, sufficient conditions in deriving the homogeneous conditions of such systems of such systems of such systems.

### **Preliminaries on Algebraic Geometry Preliminaries on Algebraic Geometry**

Considering an unmixed polynomial system whose supports are identical in the following form:  $\overline{a}$  is support to the following form:

$$
f_i \coloneqq \sum_{\alpha \in A} C_{i\alpha} x^{\alpha} \tag{1}
$$

Vol. 16 No. 2, DECEMBER 2019<br>With  $i = 1,...,n + 1$ . Let  $f_1,...,f_{n+1} \in \mathbb{C}[x_1,...x_n]$  be  $n + 1$  polynomials in With  $i = 1,...,n + 1$ . Let  $J_1,...,J_{n+1} \in \mathbb{C}[X_1,...,X_n]$  be  $n + 1$  polynomials in<br>*n* variables with the same Newton polytope  $Q \subset \mathbb{R}^n$  which is known as a convex hull of the supports,  $Q = conv(A)$  and  $A = Q \cap \mathbb{Z}^n = \{\alpha_1,...,\alpha_N\}$ convex hull of the supports,  $Q = conv(A)$  and  $A = Q \cap \mathbb{Z}^n = \{\alpha_1, ..., \alpha_N\}$ <br>is a set of the supports of the polynomials affinely span  $\mathbb{Z}^n$ . Note that R is is a set of the supports of the polynomials affinely span  $Z<sup>n</sup>$ . Note that R is<br>the set of real numbers, Z is the set of integer numbers and both sets are in the affine space  $\mathbb{C}^n$ . *n* variables with the same Newton polytope  $Q \subset \mathbb{R}$  which is known as a same  $n$  in  $\{x_i\}_{i=1}^n$  or  $\{x_i\}_{i=1}^n$ incumbers and  $\alpha$  is the set of  $\alpha$  is the sets are in the affine space  $\alpha$  . . 2, D<br>1 po ı. convex hull of the supports,  $Q = \text{conv}(A)$  and  $A = Q \cap \mathbb{Z}^n = \{\alpha_1, ..., \alpha_N\}$ . Is a set of the supports of the polynomials anthery span  $\mathbb{Z}$ . Note that  $\kappa$  is<br>the set of real numbers,  $\mathbb{Z}$  is the set of integer numbers and both sets are in  $\hat{\mathbf{C}}$  **n**  $\mathbf{C}$  **n** convex hull of the supports,  $Q = \text{conv}(A)$  and  $A = Q \cap \mathbb{Z}^* = {\alpha_1, ..., \alpha_N}$ the set of real numbers,  $\boldsymbol{\Sigma}$  is the set of integer numbers and both sets are in

The interesting type of hybrid matrix for this research is as the following: The interesting type of hybrid matrix for this research is as the following: The interesting type of hybrid matrix for this research is as  $\text{following:}$ The interesting type of hybrid matrix for this research is as the  $\sum_{i=1}^{\infty}$  unchereng supports are identical in the following form:

$$
\begin{pmatrix} Bez & Syl^* \\ Syl & \mathbf{0} \end{pmatrix} \tag{2}
$$

Where the entries of Syl and Syl\* are of Sylvester type, the entries of Bez are of Bèzout type and 0 is a square matrix of zero entries. In this paper,  $\frac{1}{2}$  and  $\frac{1}{2}$  or  $\frac{1}{2}$  and  $\frac{1}{2}$  is a square matrix of zero emries. In this paper, the application of algebraic geometry is focused only on the Bèzout block. The entries of the Bezout block are linear forms in the bracket variables Inc entries of the Bezout block are linear forms in the bracket variables<br>[uvw] defined as follows: Where the entries of *Syl* and *Syl*<sup>\*</sup> are of Sylvester type, the entries of Bez are of Bèzout type and 0 is a square matrix of zero entries. In this paper, the application of algebraic geometry is focused only on the Bezout block. bez are of Bezout type and 0 is a square matrix of zero entries. In this paper, the application of algebraic geometry is focused only on the Bèzout block. Where the entries of  $Syl$  and  $Syl^*$  are of Sylvester type, the entries of  $\int$  defined as follows: the application of algebraic geometry is focused only on the Bezout block. [uvw] defined as follows: *N* here the entries of *Syl* and *Syl*<sup>\*</sup> are of *Sylvester* type, the entries of *Bez Syl*\*

$$
[\mu \nu \gamma] = \begin{vmatrix} C_{1\mu} & C_{1\nu} & C_{1\gamma} \\ C_{2\mu} & C_{2\nu} & C_{2\gamma} \\ C_{3\mu} & C_{3\nu} & C_{3\gamma} \end{vmatrix}
$$
 (3)

Where  $C_{in}$ ,  $C_{in}$ ,  $C_{kv}$  are the coefficients  $x^{\alpha} \in f_i$  of . The bracket variables satisfy a certain set of inequalities formulated by Khetan [10] and <br>computed was computed to face and when dended with 50, named 5. Data and variables satisfy a certain set of inequalities formulated by Knetan [10] and<br>were implemented to five rules by Ahmad and Aris [9], named 5-Rule and were impremented to live rates by Animad and Aris [5], hanned 5-Kure and was computed with complexity  $\mathcal{O}(n^4)$ . inequalities formulated by Khetan [10] and were implemented to five rules by Ahmad and Aris Where  $C_{i\mu}$ ,  $C_{j\nu}$ ,  $C_{k\gamma}$  are the coefficients  $x^{\alpha} \in f_i$  of . The bracket Where  $C_{i\mu}, C_{j\nu}, C_{k\gamma}$  are the coefficients  $x^{\alpha} \in f_i$  of The bracket Where  $C \subset G$  is a coefficient of  $\mathcal{C} \subset G$  is a certain set of  $\mathcal{C}$  and  $\mathcal{C} \subset G$  and set of  $\mathcal{C}$  and set of where  $C_{ll}U_{j}U_{j}U_{k\gamma}$  are the economic  $W_{l} - J_{l}$  of fine cluster Where  $C_{i\mu}, C_{j\nu}, C_{k\gamma}$  are the coefficients  $x^{\alpha} \in f_i$  of. The bracket variables satisfy a certain set of inequalities formulated by Knetan [10] and

Rule1: 
$$
\forall k \in R_3, \gamma_k > a_k
$$
  
\nRule2:  $\exists j \in R_2, \gamma_j \le a_j$   
\nRule3:  $\forall j \in R_2, \upsilon_j + \gamma_j > a_j$   
\nRule4:  $\exists_i \in R_1, \upsilon_i + \gamma_i \le a_i$   
\nRule5:  $\forall i \in R_1, \mu_i + \upsilon_i + \gamma_i > a_i$ 

Scientific Research Journal *jjj Rule j R a Rule k R a kk Rule R a* ,:1 **POLENTIFIC IN** 

be written as follows,

Where  $R_1$ ,  $R_2$  and  $R_3$  are the similar partitions of the fan constructed in Khetan [10]. In addition, Khetan's formula of the Bèzout matrix is also considered and restated in the following theorem.  $\mathcal{L}_{\mathcal{B}}$ *Rule <sup>i</sup> <sup>R</sup>aRule <sup>R</sup>aRule <sup>j</sup> <sup>R</sup> <sup>a</sup>* ,:5,:4,:3  $\mathcal{L}$  is also considered and restated in the following theorem.  $\mathcal{L}_{\mathcal{B}}$ *i* are the similar partitions of the fan constr considered and restated in the following theorem. Where  $R_1$ ,  $R_2$  and  $R_3$  are the similar partitions of the fan constructed and the state of the  $\overline{M}$ and the contract of the contract  $R$  *Ruleiva and rest* in Khetan [10]. In addition, Khetan's formula of the Bèzo ........<br>..  $\frac{1}{2}$ is idered and restated in the following theorem.

**Theorem 1.** *The Bèzout matrix is the matrix of the linear map*   $T_Q$ : $(S_Q)^* \rightarrow S_{int(2Q)}$  defined by **Theorem 1.** The Bèzout matrix is the matrix of the linear map  $T_Q$ : $(S_Q)^* \rightarrow S_{int(2Q)}$  defined by  $TQ \cdot (QU)$  *P*  $\omega_{\text{int}}(2Q)$  *matrix*  $\omega$  *j* **Theorem 1.** The Bèzout matrix is the matrix of the lineer  $\int_{R}$   $\$  $\mathcal{L} \setminus \mathcal{L}$  matrix is also considered in the following the following the following the following the following theorem.  $H_Q \cdot (9Q) \rightarrow S_{int(2Q)}$  uchned by *Rule j R a*  $\overline{a}$  $\frac{1}{2}$ ,  $\frac{1}{2}$ 

$$
T_{Q}\left[\left(v^{\alpha}\right)^{*}\right]=\sum_{\left(\mu,\nu,\lambda\right)\in F_{\alpha}\subset A^{3}}\left[\mu\nu\gamma\right]v^{\mu+\nu+\gamma-\alpha-\omega_{0}}
$$

 $\omega_0 = (1, 1, ..., 1)$ , and  $F_\alpha$  is the set of all triples  $(\mu, \nu, \gamma) \in \phi_Q(A)^3$ **The B**izon is the Bèzout matrix is the matrix is the matrix of the matrix is the matrix of the  $\omega_0$  – (i, i,..., 1), and  $\Gamma_\alpha$  is the set of an urpres  $(\mu, \nu, \gamma) \in \varphi_Q(\Lambda)$ <br>satisfying (4).  $1 - 1$   $1 - 0$   $11 + 1$   $2 - 1$  $\overline{\mathbf{3}}$  1,,1,1 <sup>0</sup> *, and F is the set of all triples* <sup>3</sup> ,, *<sup>Q</sup> A satisfying (4).*  $_{\alpha}$  is the set  $(x, 1)$ , and  $F_\alpha$  is the set of  $\sim$  $\omega_0 = (1, 1, \dots, 1)$ , and  $F_{\alpha}$  is the set of all triples  $(\mu, \nu, \gamma) \in \phi_0(A)^3$ \* : *QQ SS <sup>Q</sup> defined by*  $\mathcal{L}^{\mathcal{A}}$  and  $\mathcal{L}^{\mathcal{A}}$  and  $\mathcal{L}^{\mathcal{A}}$  and  $\mathcal{L}^{\mathcal{A}}$ 

The bracket variables  $[\mu \nu \gamma]$  in Theorem 1, are also known as the coefficients of the facet variables for the polynomial ring and the exponent vectors of the facet variables as follows, The bracket variables  $\mu v \gamma$  in Theorem 1, are also known as the coefficients of the facet variables for the polynomial ring and the exponent vectors of the facet variables as follows vectors of the facet variables as follows, The bracket variables  $[\mu \nu \gamma]$  in Theorem 1, are also known as the coefficients of the facet variables for the polynomial ring and the exponent The bracket variables  $[\mu \nu \gamma]$  in Theorem 1, are also known<br>coefficients of the facet variables for the polynomial ring and the exp  $\frac{1}{2}$  bectors of the facet variables as follows vectors of the facet variables as follows,  $\overline{a}$ 1, netficients of the facet variables for the polynomial ring and the exponent

$$
\phi_{Q}\big(x^{\alpha}\big) = y^{\alpha} = \prod_{i=1}^{s} y_i^{\phi_{Q}(\alpha)}.
$$

From the formulation in Theorem 1, it will produce a homogeneous From the formulation in Theorem 1, it will produce a homogeneous<br>polynomial, and the bracket  $[\mu \nu \gamma]$  is denoted as coefficients of facet variables  $y_p$ ,  $y_p$ ,  $y_s$  while  $\mu + \nu + \gamma - \alpha - \omega_0$  is the formula to enumerate the exponents for each facet variable. Sum of the exponents must equal for the exponents for each facet variable. Sum of the exponents must equal for each monomial of the homogeneous polynomials with bracket coefficients. Therefore, one can be written as follows,  $\mathbf{F}$  is the formulation in Theorem 1, it will produce a homogeneous polynomial, and the formulation  $\mathbf{F}$  <sup>0</sup> , *iiiiii <sup>Q</sup> <sup>i</sup> a* for ,,1 *si* . From the formulation in Theorem 1, it will produce a homogeneous From the formulation in Theorem 1, it will produce a homogeneous<br>polynomial, and the bracket  $[\mu \nu \gamma]$  is denoted as coefficients of facet<br>veribles used in while  $\mu + \nu + \gamma \gamma$  and it is formula to enumerate  $\overline{\mathbf{a}}$   $\overline{\mathbf{a}}$   $\overline{\mathbf{b}}$   $\overline{\mathbf{b}}$   $\overline{\mathbf{b}}$  $\mu$  formula to enter the exponent  $\mu$  formula to exponent sum of the exponents must exponent of the exponents must equal to exponent the exponents must exponent to the exponents must exponent the exponents must exponent variables  $y_p$ ,  $y_p$ ,  $y_p$  while  $\mu + \nu + \gamma - \alpha - \omega_0$  is the formula to enumerate  $\overline{\phantom{a}}$  $\omega$  *i*  $\omega$  *i*  $\omega$  *i*  $\omega$  *i*  $\omega$  *j*  $\omega$  each monomial of the homogeneous polynomials with bracket coefficients.

for each monomial of the homogeneous polynomials with bracket coefficients. Therefore, one can

$$
\phi_Q(\alpha)_i = \langle \alpha, v_i \rangle + a_i \cong \mu_i + v_i + \gamma_i - \alpha_i - \omega_0 \text{ for } i = 1, ..., s.
$$

In the next definition is to define projective space. This space is very in the hext definition is to define projective space. This space is very<br>important for homogeneous polynomial system. In the next definition is to define projective space. This space is very important for  $\mathcal{L}_\mathcal{A}$  $\frac{1}{2}$  the next definition is to definition is very important for  $\frac{1}{2}$ 

$$
P^{n}(K) = (K^{n+1} - {0})/\sim
$$

**Definition 1**. [7] Given the set of *n*-dimensional projective space where each **Definition 1.** [*I*] Given the set of *n*-dimensional projective space where each nonzero  $(n + 1)$  - tuple,  $(x_0,...,x_n) \in K^{n+1}$  defines a point  $p \in P^n(K)$  and nonzero  $(n + 1)$  - tuple,  $(x_0,...,x_n) \in K^{n+1}$  aefines a<br> $(x_0,...,x_n)$  is called the homogeneous coordinate of p. **Definition 1.** [7] Given the set of *n*-dimensional projective space where each  $\sum_{i=1}^{n}$ nonzero  $(n + 1)$ -tuple,  $(x_0,...,x_n) \in K^{n+1}$  defines a point  $p \in \mathbf{P}^n(K)$  an **Definition 1.** [7] Given the set of *n*-dimensional projective space where each  $\left(\alpha_0, \ldots, \alpha_n\right)$  is cancel are nomogeneous coordinate  $\alpha$  $(x_0, \ldots, x_n)$  is called the homogened

Geometrically the polynomial (1) can be viewed as the Newton polytope  $Q \subset \mathbb{R}^n$  and support  $A = Q \cap \mathbb{Z}^n = \{\alpha_1, ..., \alpha_N\}$ . The support can be plot in a Cartesian coordinate,  $(x, y)$  if the dimension of affine space is 2 and it is called polytope. Thus, in this implementation, the polynomial equations with two variables and three equations are used. Polytopes have faces, with two variables and timed equations are used. Tolytopes have faces, edges and vertices. The faces of polytope are when polygons lying in plane, while edges are line segments that connect certain pairs of vertices (faces while edges are line segments that connect certain pairs of vertices (faces<br>of dimension 1) and vertices are points (faces of dimension 0). Hence, each facet  $F = \{r_1, ..., r_s\}$  of Q is a polytope of dimension less than dimension of Q. Therefore, if Q has dimension *n*, then facets are faces of dimension  $n-1$ . edges and vertices. The faces of polytope are when polygons lying in plane,  $t = \frac{1}{\sqrt{2}}$  and vertices  $\frac{1}{\sqrt{2}}$  and vertices are points (faces of dimension 1) and vertices are points (faces of dimension 1) and vertices are points (faces of dimension 1) and vertices are points (faces of dimen  $\chi$ , therefore,  $\pi \chi$  face dimension *m*, shen have a are have of dimension  $n = 1$ . while edges are line s of dimension 1) and ve  $\begin{array}{c}\n\text{d} \text{det} \ \mathbf{F} = \{i_1, \ldots, i_s\} \ \text{d} \text{d} \\
\text{f} \ \text{f} \ \text{d} \text{h} \text{d} \text{e} \\
\text{f} \ \text{d} \text{h} \text{e} \\
\text{$ of *Q*. Therefore, if *Q* has dimension *n*, then facets are faces of dimension *n* – 1.

To define a face of an arbitrary Newton polytope, an affine hyperplane is needed. Let  $v$  be a nonzero vector in  $\mathbb{R}^n$ , an affine hyperplane is defined by an equation of the form  $m \cdot v = -a$ . Therefore, for every unmixed system of polynomial equations, there exists an associated Newton polytope *Q* defined by defined by polytope *Q* defined by  $\sigma$ y an equation of the form  $m : v = a$ . Therefore, for every unimated system of polynomial equations, there exists an associated Newton polytope Q of polynomial equations, there exists an associated Newton polytope  $Q$ <br>defined by  $T$  and  $T$  every unique system of polynomial equations, there exists an associated Newton and Aesters and Newton  $T$ Therefore, for every unmixed system of polynomial equations, there exists an associated Newton and associated N

$$
a_i = -\min_{m \in \mathcal{Q}} \langle m \cdot \nu \rangle \tag{5}
$$

Thus, any Newton polytope Q in  $\mathbb{R}^n$  with s number of edges can be defined<br>by its facet inequalities given by by its facet inequalities given by inequalities given by

$$
Q = \left\{ m \in R^n \mid \langle m, v_i \rangle \ge -a_i, i = 1, \dots, s \right\},\tag{6}
$$

for some integers  $a_1, ..., a_s$  which are referred to as the data for Q. The inner normal  $v_1, ..., v_s$  are called rays. The set of data and inner normal are unique formal  $v_1, ..., v_s$  are called tays. The set of data and limer hormal are and de-<br>for the facets. Hence the following definition are needed for homogenisation coordinate  $x^{a_1}$ ,.., <sup>as</sup>. coordinate  $x$   $, \ldots$   $, \ldots$ 

**Definition 2.** [10] *The Q-homogenisation map*  $\phi$ <sub>O</sub> :  $\mathbb{Z}^n \to \mathbb{Z}^s$  *is defined by for*  $i = 1,...,s$ . *<sup>X</sup> <sup>s</sup>* ,, *yyKS* <sup>1</sup> *such that the monomials are graded.* **Definition 2.** [10] *The Q-homogenisation map sn <sup>Q</sup>* : **ZZ** *is defined by <sup>Q</sup> <sup>i</sup> aii* , *for*  $\textit{for } i = 1,...,S$ .  $for i = 1,...,s$ . **Definition 2.** [10] The Q-homogenisation map  $\phi_Q : \mathbb{Z}^n \to \mathbb{Z}^s$  is defined by

**Definition 3.** [10] *The homogeneous coordinate ring for*  $X = X_A$  *is the* **Definition 3.** [10] *The homogeneous coordinate ring for*  $X = X_A^A$  *is the polynomial ring*  $S_X = K[y_1, ..., y_s]$  such that the monomials are graded. *Q* determining the toric variety *X <sup>A</sup>* . **Definition 3.** [10] *The homogeneous coordinate ring for XX <sup>A</sup> is the polynomial ring Demittion 3.* [10] *The nomogeneous coor*  $X = \{y_1, \ldots, y_g\}$  and the monomials are graded. **Definition 3.** [10] *The no*lynomial ring  $S = k$ 

Let *S* be the polynomial ring,  $S = C[y_1, ..., y_s]$  with one variable for each ray in the fan  $\Sigma Q$  determining the toric variety  $X_A$ . Let S be the polynomial ring,  $S = C[y_1,...]$ Let *S* be the polynomial ring, *<sup>s</sup>* ,, *yyS* **C** <sup>1</sup> with one variable for each ray in the fan

SCIENTIFIC RESEARCH JOURNAL  $\overline{\text{SCIEN}}$ *Q* determining the toric variety *X <sup>A</sup>* . *Q* determining the toric variety *X <sup>A</sup>* . Let *S* be the polynomial ring, *<sup>s</sup>* ,, *yyS* **C** <sup>1</sup> with one variable for each ray in the fan *Q* determining the toric variety *X <sup>A</sup>* .

Let  $Q \subset \mathbb{R}^n$  be the Newton polytope of dimension *n*, and let the support  $A = \{\alpha_1, ..., \alpha_N\}$  such that convex A equals Q. The toric variety  $\overline{a}$  $E_A[(C^{\dagger})^{\dagger}] \subset P^{n-1}$  is the dimension n variety defined as the Zariski closure of the set  $\{x^{\alpha_1},...,x^{\alpha_N}\}\)$  that is the image of  $\mathcal{P}_A$  where  $\alpha_i \in \mathbb{Z}^n$  and  $x = (x_1,...,x_n) \in (\mathbb{C}^*)^n$ . Precisely the elements of  $\mathcal{P}_A$  are described in the following. following, support  $A = \{a_1, ..., a_N\}$  such that convex *A* equals *Q*. The toric variety  $X = \phi \left( \left( e^{*\gamma} \right)^n \right) \in \mathbf{P}^{N-1}$  . At a convex *A* equals *Q*. The toric variety  $\alpha$   $\alpha$   $\beta$   $\beta$   $\alpha$   $\beta$   $X_A = \overline{\phi_A((c^*)^n)} \subset P^{N-1}$  is the dimension n variety defined as the Zariski closure of the set  $\{x^{\alpha_1},...,x^{\alpha N}\}\)$  that is the image of  $\phi_A$  where  $\alpha_i \in \mathbb{Z}^n$  and  $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$ . Precisely the elements of  $\mathcal{A}$  are described in the Precisely the elements of *<sup>A</sup>* are described in the following, Precisely the elements of *<sup>A</sup>* are described in the following, Precisely the elements of *<sup>A</sup>* are described in the following,  $\mathcal{A} = \{ \alpha_1, ..., \alpha_N \}$  such that convex *A* equals *Q*. The toric variety  $X_A = \phi_A[(C')] \subset P^{N-1}$  is the dimension *n* variety defined as the Zariski closure of the set  $\{X_1, \ldots, X_n\}$  that is the image of  $\{A_n\}$  where  $\alpha_i \in \mathbb{Z}$  and *<sup>N</sup> <sup>n</sup> A <i>A A <i>A <i>A <i>A <i>A <i>A <i>A <i>A <i>A <i>A <i>A*  $d_{\text{per}}(x) = (a_1, \ldots, a_N)$  such that convex  $A$  equals  $g$ . The toric variety ,, <sup>1</sup> that is the image of *<sup>A</sup>* where *<sup>n</sup> <sup>i</sup>* **Z** and  $\int \alpha_1$   $\alpha N$   $\int$ *n*

$$
A \xrightarrow{\phi_A} \mathbf{P}^{N-1}
$$
  
\n
$$
\alpha_1 \longrightarrow x^{\alpha_1}
$$
  
\n
$$
\alpha_2 \longrightarrow x^{\alpha_2}
$$
  
\n
$$
\vdots
$$
  
\n
$$
\alpha_N \longrightarrow x^{\alpha_N}
$$

For all facets, the set of cones  $\sigma_{\tau}$  are generated by inner normals. Then  $\sum_{i}^{\infty}$   $\sigma_{\tau}$  |  $\tau$  is a face of *Q* } is the normal fan of *Q*. This gives a toric Then  $\sum_{Q} \frac{1}{\sigma_r} |\tau|$  is a face of Q } is the normal fan of Q. This gives a toric<br>variety denoted  $X_A$ . Each vertex of the cones is spanned by the inner normal<br>v. corresponding to foots (odges for two dimension) which variety denoted  $A_A$ . Each vertex of the cones is spanned by the liner horman<br> $v_i$  corresponding to facets (edges for two dimension) which are incident to  $v_i$  corresponding to facets (edges for two dimension) which are incident to the vertex. The characterisation of the normal fan is stated in the following theorem.  $\mu$  the order. The vertex the characterization of the normal fan is stated in the following in the fo the vertex. The characterisation of the normal fan is stated in the following  $\alpha$  the order. The characterization of the normal fan is stated in the following in  $\beta$  denoted  $X_{\overline{A}}$ . Each vertex of the cones is spanned by the inner normal **Theorem 2.** [4] *The normal toric variety of a fan* Then  $\sum_{\bar{c}} [\sigma_{\bar{c}} | \tau]$  is a face of *Q* } is the normal fan of *Q*. This gives a toric variety denoted  $X_A$ . Each vertex of the cones is spanned by the inner normal x. The characterisation of the normal fan is stated in the following  $\mu$  are incident to the characterization of the characterization of the characterization of the state  $\mu$ *x*  $\mathbf{r}$  $\sum_{i=1}^{n}$ an M Ĩ.  $\sim$  200  $\mu$  *<sup>Q</sup> <sup>Q</sup>* <sup>|</sup> is <sup>a</sup> faceof is the normal fan of *<sup>Q</sup>*. This gives a toric variety denoted *<sup>X</sup> <sup>A</sup>* . Each vertex

> **Theorem 2.** [4] The normal toric variety of a fan  $\sum_{\mathcal{Q}} \in \mathbb{R}^n$  is projective if and only if  $\Sigma Q$  is the normal fan of an n-dimensional lattice polytope in  $\mathbf{R}^n$ . **eorem 2.** [4] The normal force variety of a fan  $\sum_{\varrho}^{\infty}$  is profective if and

Besides a complete characterisation of polytope  $Q$  in terms of the rays in its normal fan, Weil divisors are describing in the following proposition. Besides a complete characterisation of polytope  $\mathcal Q$  in terms of the rays Besides a complete characterisation of polytope  $Q$  in terms of the rays **Proposition 3.**  $\frac{1}{2}$  *b*  $\frac{1}{2}$  on the  $\frac{1}{2}$  on  $\frac{1}{2}$  on  $\frac{1}{2}$ 

**Proposition 3.** [10] The  $v_i$  are in one to one correspondence with the T-invariant prime Weil divisors on  $X_A$ . Di denotes the divisor corresponding to  $v_i$ . **oposition 3.** [10] The v are in one to  $\theta$ 

A divisor  $D = \sum_{i} a_i D_i$  determines a convex polytope of (6). The divisor is best described as a line bundle and a global section of that line bundle. Adivisor  $D - \sum_{i=1}^{n_i}$  determines a convex polytope of (6). The divisor<br>is best described as a line bundle and a global section of that line bundle. A divisor D =  $\sum_{a_iD_i}$  determines a convex polytope of (6). The divisor is best described a convex points a convex political and  $\alpha$ 

The following example is to show an edge and its inward normal of a Newton polytope in  $\mathbb{R}^2$  that defines a hyperplane  $\langle m, v \rangle = -a$ .

**Example 1.** Consider a face of Newton polytope in  $\mathbb{R}^2$  given in Figure 1. **Example 1.** Consider a face of Newton polytope in  $\mathbf{R}^2$  given in Figure 1.<br>The edge AB joins the vertex A (0,1) and B (1, 2), defines a hyperplane  $\langle m, v \rangle = -a$  where *m* is any point on *AB* and  $v = (1, -1)$  is an inward normal to *AB*. 1,1 is an inward normal to *AB*.



 **Figure 1: An Edge an Inward Normal**

Since *m* lies on *AB*, *m* (*x*, *x* + 1). Therefore  $\langle m, v \rangle = \langle (x, x+1), (1, -1) \rangle =$ *x* - *x* - 1 = - 1. Here *a* = 1. For instance,  $\frac{1}{2}, \frac{3}{2}$  is on *AB* giving  $\left\langle \frac{1}{2}, \frac{3}{2} \right\rangle (1, -1)$  = -1. In addition, any point any point *m* ' in the shaded region satisfies the equation  $\langle m', v \rangle > -1$ . <sup>1</sup> is on *AB* giving 11,1, <sup>2</sup>  $\overline{1}$  $\mathbf{e} a = \mathbf{I}$ <br>nint any  $\overline{a}$ ć  $\mathbf{a}$  $a = 1$ . For instance,  $\left(\frac{1}{2}, \frac{1}{2}\right)$  is on *AB* giving the any point *m*' in the shaded region satisfies 1. For instance,  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is on *AB* giving  $\left\langle \left(\frac{1}{2}, \frac{3}{2}\right), (1, -1)\right\rangle = -1$ .<br>Iny point *m* ' in the shaded region satisfies the equation  $\langle m, v \rangle > -1$ . ſ 2  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is on *AB* giving  $\left\langle \left(\frac{1}{2}, \frac{3}{2}\right) (1, -1) \right\rangle = -1$ .  $\lim_{n \to \infty}$  point *m* 11, the shaded region satisfie instance,  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is o Ì is on *AB* giving  $\left\langle \left( \frac{1}{2}, \frac{3}{2} \right), (1, -1) \right\rangle = -1$ . x - x - 1 = - 1. Here  $a = 1$ . For instance,  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is on *AB* giving  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is on *AB* giving  $\left(\frac{1}{2}, \frac{3}{2}\right)$ 'n Since *m* lies on *AB*, *m* (*x*, *x* + 1). Ineference  $\langle m, v \rangle = \langle (x, x+1), (1, -1) \rangle -$ <br>  $c - x - 1 = -1$ . Here  $a = 1$ . For instance,  $\left(\frac{1}{2}, \frac{3}{2}\right)$  is on *AB* giving  $\left\langle \left(\frac{1}{2}, \frac{3}{2}\right), (1, -1) \right\rangle = -1$ . In addition, any poi  $\ddot{\phantom{0}}$  $\lim_{x \to 0} \lim_{x \to 0} \lim_{x$ shaded region satisfies the equation *m* 1, .

#### **Implementation Of The Bèzout Block Implementation Of The Bèzout Block** Considering 0,0 as an exponent vector (the constant term being non zero), a **Implementation Of The Bèzout Block**  $\sum_{i=1}^n$  and below as an exponent vector (the constant term being non  $\sum_{i=1}^n$  $\frac{1}{2}$  at the unmixed system worked by Khetan  $\frac{1}{2}$ **IMP**

Considering  $(0,0)$  as an exponent vector (the constant term being non zero), a generalisation for the unmixed system worked by Khetan [10], we preserved the geometric structure, gives the system: generalisation for the unmixed system worked by Khetan [10], we preserved the geometric Considering  $(0,0)$  as an exponent vector (the constant term being non

$$
f_i = C_{i1}x^0y^0 + C_{i2}x^h y^0 + C_{i3}x^0y^k + C_{i4}x^h y^k + C_{i5}x^{2h} y^k + C_{i6}x^h y^{2k}
$$
 (7)

with  $i = 1,2,3$ , and A. {(0, 0), (h, 0), (0, k), (h, k), (2h, k), (h, 2k)} Geometrically, by Example 1, the Newton polytope for the unmixed polynomial system (7) can be illustrated as in Figure 2.



**Figure 2: Translated Newton Polytope of System (7) Figure 2: Translated Newton Polytope of System (7) Figure 2: Translated Newton Polytope of System (7)**

In this paper, the application of algebraic geometry is discussed with respect to the computation of all valid combinations of  $(\mu, \nu, \lambda)$  that satisfy the inequalities in 5-Rule (4) and to compute the respective degree of the facet variables defined by the polynomial equations (7). the inequalities in 5-Rule (4) and to compute the respective degree of the **Figure 2. Example 2. Properties of System (7)** In this paper, the application of all valid combinations of  $(\mu, \nu, \lambda)$ 

The vertices of the Newton polytope *Q* shown in Figure 2 are sorted in counter clockwise, starting at the origin  $(0, 0)$  as a distinguish of point. The direction of sorting the vertices is fixed to obtain an appropriate pair of a vertex and its ray which gives the hyperplane data computation of the So a vertex and its ray which gives the hyperplane data computation of the<br>Newton polytope. The hyperplane data is written as  $a_1, ..., a_s$  and is defined<br>by the equation of hyperplane (5) so that  $D = \sum a_i D_i$  is the correspond Newton polytope. The hyperplane data is written as  $a_1, ..., a_s$  and is defined<br>by the equation of hyperplane (5), so that  $D_0 = \sum a_i D_i$  is the corresponding<br>divisor. These divisors are the elements of the vector space S divisor. These divisors are the elements of the vector space  $S_Q$ . The vertices of the Newton polytope *Q* shown in Figure 2 are sorted in counter clockwise, The vertices of the Newton polytope  $Q$  shown in Figure 2 are sorted in counter clockwise, starting at the origin  $(0, 0)$  as a distinguish of point.

Suppose  $h = k = 1$ , the computation of the partition of the fan,  $R_1, R_2$ and  $R_3$  are derived as in Table 1. Thus, the sets of partition of the fans are and  $R_3$  are derived as in Table 1. Thus, the sets of partition of the Tails are<br> $R_1 = \{1, 5\}, R_2 = \{2, 3\}$  and  $R_3 = \{4\}.$  Each vertex of the Newton polytope gives a two-dimensional cone in the normal fan where the rays,  $v$  is the generator of the cones. By gluing of cones, the resulting normal fan of the translated Newton polytope (Figure 2) is shown in Figure 3. This fan gives relatively to the property of the set of  $\sigma_1$ ,  $\sigma_2$ ,  $\sigma_3$ ,  $\sigma_4$ ,  $\sigma_5$  are generated<br>by the bases of  $\mathbb{Z}^2$ . This implies that the offine toric verieties  $Y$ , are conjected **P**, such that all two-dimensional cones  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$  are generated<br>by the bases of **Z**<sup>2</sup>. This implies that the affine toric varieties  $X_{\sigma_i}$  are copies of  $\mathbb{C}^2$ . By gluing together five copies of  $\mathbb{C}^2$ ,  $\mathbb{P}^4$  is constructed. For simplicity, each ray is represented by numbers 1, 2, 3, 4, 5. Hence, the partitions of the by the bases of  $\mathbb{Z}^2$ . This implies that the affine toric varieties  $X_{\sigma_i}$  are copies of  $C^2$ . By gluing together five copies of  $C^2$ ,  $P^2$  is constructed. For simplicity, each ray is represented by numbers 1, 2, 2, 4, 5. Hence, the pertition cash fay to represented by numbers  $1, 2, 3, 1, 2, 1$  is repair.

fan for the case  $h = k = 1$  is viewed in Figure 3 by applying the concept of cones and fan.

Partition $R_i$		Inward normal $v_i$
$R_1 = \{i : v_i = c_1v_1 + c_2v_2\}$		$\begin{array}{c} v_1 = c_1(1,0) + c_2(0,1) = (1,0) \\ c_1 = 1, \ \ c_2 = 0 \\ v_5 = c_1(1,0) + c_2(0,1) = (1,-1) \\ c_1 = 1, \ \ c_2 = -1 \end{array}$
with $c_1 \ge 0$ and $c_2 \le 0$		
$R_2 = \{i : v_i = c_1v_1 + c_2v_2\}$		$V_2 = c_1(1,0) + c_2(0,1) = (0,1)$ $c_1 = 0, c_2 = 1$
with $c_1 \le 0$ and $c_2 \ge 0$		
		$v_3 = c_1(1,0) + c_2(0,1) = (-1,1)$ $c_1 = -1, c_2 = 1$
$R_3 = \{i : v_i = c_1v_1 + c_2v_2\}$	4	$V_4 = c_1(1,0) + c_2(0,1) = (-1,-1)$ $c_1 = -1, c_2 = -1$
with $c_1 < 0$ and $c_2 < 0$		

**Table 1: Partition of the Fan** *Ri*  **Table 1: Partition of the Fan** *Ri* **Table 1: Partition of the Fan** *R* 



*Figure* **3: The Fan of Figure 2 When**  $h = k = l$ 

The above fan is determined by the divisor  $D = \sum_{i=1}^{n} a_i D_i = D_3 + D_4$ *3D<sub>4</sub>* + *D<sub>5</sub>* such that  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_3 = 1$ ,  $a_4 = 3$ ,  $a_5 = 1$  and generates by  $v_1, v_2, v_3, v_4, v_5$ . The above fan is determined by the divisor  $D = \sum a_i D_i = D_i +$  $2D + D$  such that  $a = 0$ ,  $a = 0$ ,  $a = 1$   $a = 3$ ,  $a = \overline{I}$  and generates by  $p_1D_1 + D_2$  such that  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_3 = 1$ ,  $a_4 = 3$ ,  $a_5 = 1$  and generates by above fan is determined by the divisor  $D = \sum_{i=1}^{5} a_i D_i = D_3 + \cdots$  $a_1 + D_5$  such that  $a_1 = 0$ ,  $a_2 = 0$ ,  $a_3 = 1$ ,  $a_4 = 3$ ,  $a_5 = 1$  and  $T_A$   $\sim$  5-Rule involves combinatorial approach where all the nonzero all th SCIENTIFIC RESEARCH JOURNAL **1** *i*  $\overline{D}$  3  $\overline{$ *<sup>i</sup>*

The formulation of the 5-Rule involves combinatorial approach where all the nonzero coefficients in the polynomials are considered and it applied to the structure of the Newton polytope of the associated polynomial system  $(7)$ . In order to obtain the efficiency and effectiveness in generating the bracket variables  $[\mu v \gamma](3)$  for the entries of the Bèzout block, we formulated the new conditions for 5-Rule, as follows:  $\mathcal{L}_{\text{max}}$ 

The bracket variables are rejected if,  $T_{\rm eff}$  because are regelected if, we are regular to the contract of  $T_{\rm eff}$  because  $T_{\rm eff}$  are rejected if, and  $T_{\rm eff}$  $\frac{1}{1}$ 

- $(\mu = \nu)$  or  $(\nu = \gamma)$  or  $(\mu = \gamma)$  and The bracket variables are rejected if,<br>  $\bullet$  ( $\mu = \nu$ ) or ( $\nu = \nu$ ) or ( $\mu = \nu$ ) and , *Rule5* , *Rule4* . , *Rule5* , *Rule4*.
	- $\bullet$   $(\mu, \nu) \in$  Rule5  $\neq (\mu, \nu) \in$  Rule4.

Applying those conditions in formula (4), the row elements are Applying those conditions in formula (4), the fow elements are<br>uniquely defined, and the algorithm terminates with the correct matrix dimension. algorithm terminates with the correct matrix dimension dimension. The row elements in formula (4), the row elements are uniquely defined, and the ro uniquely defined, and the algorithm terminates with the correct matrix  $\theta$ and algebraic geometry, a systematic geometry, a systematic approach for a systematic a

Based on the concepts of divisors and algebraic geometry, a systematic approach for homogenising the polynomial equations have been realised and designed. Therefore, by Definition 2, the homogeneous coordinate for each support of system (7) is constructed for the case  $h = k = 1$ , Based on the concepts of divisors and algebraic geometry, a systematic , *Rule5* , *Rule4* . approach for nonogenising the polynomial equations have been realised and designed. Therefore, by Definition 2, the nonfogeneous coordinate for<br>each sumport of system (7) is constructed for the case  $h = k = 1$ Based on the concepts of divisors and algebraic geometry, a systematic<br>approach for homogenising the polynomial equations have been realised  $\mathbf{u}$  and  $\mathbf{v}$  with the correct matrix dimension. and designed. Therefore, by Definition 2, the homogeneous coordinate for each support of system (*i*) is constructed for the case  $n = k = 1$ ,



of the hybrid resultant matrix. In (4), given that  $k \in R$ ,  $j \in R$  and  $i \in R$ . The bracket variables are constructed. Starting with the first exponent, (0,<br>The bracket variables are constructed. Starting with the first exponent, (0,  $(0)$  we have, These homogenisation coordinates are in  $S_{\text{int}(2Q)}$  and index the rows 0) we have, 0) we have, of the hybrid resultant matrix. In (4), given that  $k \in R_{3}$ ,  $j \in R_{2}$  and  $i \in R_{1}$ .<br>The bracket variables are constructed. Starting with the first exponent, (0,<br>0) we have

$$
(0,0) \rightarrow 0 \quad 0 \quad 1 \quad 3 \quad 1
$$
  

$$
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
$$
  

$$
\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5
$$

Continuing 5-Rule process, the empirical results for the combinations  $\overline{a}$  $(\mu, \nu, \gamma)$  of all points in Q, are shown in Table 2. Definition 3 is used on the construction of the exponent vectors of the facet variables. The algorithm terminates with the correct matrix dimension and the computational cost is  $\mathcal{O}(n^3)$ .  $(\mu, \nu, \gamma)$  or an points in  $\zeta$ , are shown in Table 2. Definition 5 is used on the construction of the exponent vectors of the facet variables. The algorithm  $(\mu, \nu, \gamma)$  of all points in Q, are shown in Table 2. Definition 3 is used on the cost is  $\epsilon$ 

$\alpha$	$(\alpha_1,\ldots,\alpha_5)$	$(\mu, \nu, \gamma)$
(0, 0)	(0, 0, 1, 3, 1)	l 0
(1, 0)	(1, 0, 0, 2, 2)	$(2, 4, 1), (2, 6, 1), (5, 3, 1), (5, 4, 1), (5, 6, 1)$
(0, 1)	(0, 1, 2, 2, 0)	l 0
(1, 1)	(1, 1, 1, 1, 1)	$(2,6,1)$ , $(4,6,1)$ , $(5,6,1)$ , $(2,4,3)$ , $(5,4,3)$ , $(2,6,3)$ , (5,6,3)
(2, 1)	(2, 1, 0, 0, 2)	$(2,5,3)$ , $(5,6,1)$ $(5,3,4)$ $(2,4,6)$ , $(5,6,2)$ , $(5,6,4)$
(1, 2)	(1, 2, 2, 0, 0)	(2,6,3), (4,6,3), (5,6,3)

 $\boldsymbol{\tau}$  Table 2: The Combination  $\boldsymbol{\mu}, \boldsymbol{v}, \boldsymbol{\gamma} \;$  for all  $\boldsymbol{a} \in \boldsymbol{Q}$  when  $\boldsymbol{h}$  =  $\boldsymbol{k}$  = 1 Table 2: The Combination  $\mu, v, \gamma$  for all  $\alpha \in Q$  when  $h = k = 1$ 

### **Main Results**

The results on the implementations of the hybrid Sylvester-Bèzout resultant matrix are generalised. The main theorem that established the conditions that can give a determinantal hybrid resultant formula for the class of unmixed bivariate polynomial systems is derived and proven.

Let *r, t* be non-negative integers and *h*, *k* be positive integers. Consider the unmixed polynomial system of three equations in two variables of the form,

$$
f_i = C_{i1} x^r y^t + C_{i2} x^{r+h} y^t + C_{i3} x^r y^{t+k} + C_{i4} x^{r+h} y^{t+k} + C_{i5} x^{r+2h} y^{t+k}
$$
 (8)

with  $i = 1,2,3$  and the support of the system is

$$
A = \{ (r,t),(r+h,t),(r,t+k),(r+h,t+k),(r+2h,t+k),(r+h,t+2k) \}.
$$

If  $h = k$ , the system (8) whose support is reduced to  $\mathcal{L}$  because  $\mathcal{L}$ *a <i>r*,*t <i>r*<sub></sub>,*r*</sub> *<i>n*,*t <i>r*<sub></sub>,*t*</sub> *<i>n*,*t*<sub></sub> *<i>n*,*t*<sub></sub> *<i>n*,*t*<sub></sub> *<i>n*, *<i>n*, *<i>n*, *<i>n*</sup>, *a*, *i*, *a* polynomial system  $\Omega$  whose support is reduced to e system (8) whose support is reduced to  $\epsilon$  system ( $\delta$ ) whose support is reduced to

 $A = \{(r,t), (r+h,t), (r,t+k), (r+2h,t+k), (r+h,t+2k)\}.$  $A = \{(r, t), (r + n, t), (r, t + K), (r, t + 2n, t + K), (r + n, t + 2K)\}.$ *A r*,*t*,*r h*,*t*,*r*,*t k* ,*r h*,*t k* ,*r* 2*h*,*t k* ,*r h*,*t* 2*k* .  $A = \{(r,t), (r+h,t), (r,t+k), (r+2h,t+k), (r+h,t+2k)\}.$ If *h k* , the system (8) whose support is reduced to **Main Results** *<sup>i</sup> <sup>r</sup> <sup>h</sup> <sup>t</sup> <sup>k</sup>*  $= \{ (r t) (r+h t) (r t+k) (r+2h t+k) (r+h t+2k) \}$  $= \{(r,t), (r+h,t), (r,t+k), (r+2h,t+k), (r+h,t+2k) \}.$ polynomial system of three equations in two variables of the form, *<sup>i</sup> <sup>r</sup> <sup>h</sup> <sup>t</sup> <sup>k</sup>*  $\frac{1}{2}$   $\frac{1}{2}$  *<sup>i</sup> <sup>r</sup> <sup>h</sup> <sup>t</sup> <sup>i</sup> <sup>r</sup> <sup>t</sup> <sup>i</sup> <sup>i</sup> f C x y C x y C x y C x y C x y C x y*  $(1, 2, 3, 4, 5, 6, 7, 7)$  $= \{ (r t) (r+h t) (r t+k) (r+2h t+k) (r+h t+2k) \}$ 

Observing the results of the implementation gives the following theorem. with *i*  $1, 2, 3, 4$ , in 1, 3 and the system is the system is the system in the system is the system in the system is the system of the system in the system of the system in the system of the system of the system in the s e results of the implementation gives the following theorem. *A r*,*t*,*r h*,*t*,*r*,*t k* ,*r h*,*t k* ,*r* 2*h*,*t k* ,*r h*,*t* 2*k* .  $\mathbf{r}$   $\mathbf{$ of the impremements gives the renewing increase. 1 are imprementation gives the fortcoming theorem. e results of the implementation gives the following theorem.

**Theorem 4.** Let r,  $t \ge 0$ , where r,  $t \in \mathbb{Z}$  and let  $h \in \mathbb{Z}^+$ . Let  $f_1, f_2$  and  $f_3$  be an unmixed polynomial system with support  $A = \{(r,t),(r+h,t),(r,t+h),(r+h,t)\}$  $h, t+h$ ,  $(r+2h, t+h)$ ,  $(r+h, t+2h)$ }. For each  $i = 1,...,6$  the homogeneous  $h, h, h, (h + 2h, h + h), (h + h, h + 2h),$  For each  $h = 1,...,0$  the homogeneous<br>coordinate of  $e_i$  for  $r = t = 0$  equals the homogeneous coordinate of  $e_i$  for any  $r > 0$  or  $t > 0$ .  $\frac{f(x)}{f(x)} = \frac{f(x)}{f(x)}$  *equals to the homogeneous containers*  $n, t + n$ ,  $(r + 2n, t + n)$ ,  $(r + n, t + 2n)$ . For each  $t - 1$ ,..., o the homogeneous<br>coordinate of  $e_i$  for  $r = t = 0$  equals the homogeneous coordinate of  $e_i$  for  $\frac{f}{f}$  is  $\frac{f}{f}$  if  $\frac{f}{f}$  *or t*  $\geq 0$ .  $2y$  *t*  $>$  0 or  $i$   $>$  0.  $u_n$  and  $u_n$ ,  $v_n$  and  $u_n$  is the wind support  $A = \{(r, t), (r + n, t), (r, t + n), (r + n, t + h), (r + 2h, t + h), (r + h, t + 2h)\}$ . For each  $i = 1,...,6$  the homogeneous coordinate of  $e_i$  for  $r = t = 0$  equals the homogeneous coordinate of  $e_i$  for Let r,  $t \geq 0$ , where r,  $t \in \mathbb{Z}$  and let  $h \in \mathbb{Z}^+$  Let  $f_1, f_2$  and  $f_3$  be an  $T$  results on the implementations of the hybrid  $S$ Let r,  $t \ge 0$ , where r,  $t \in \mathbb{Z}$  and let  $h \in \mathbb{Z}^+$ . Let  $f_1, f_2$  and  $f_3$  be an  $p > 0$  $\mathbf{r}_i$  for the comparison components coordinate by  $\mathbf{r}_i$  for  $\mathbf{r}_i$ . point  $\sum_{i=1}^{n} f_i = 0$ , where  $f_i$  ( $\sum_{i=1}^{n} f_i = 1$  and  $f_i = 2$  and  $f_i = 2$  and  $f_i = 3$  and  $f_i = 4$  and  $f_i = 3$  and  $f_i = 1$  and  $f_i = 2$  and  $f_i = 3$  and  $f_i = 1$  and  $f_i = 2$  and  $f_i = 3$  and  $f_i = 1$  and  $f_i = 2$  and  $f_i = 3$  $m, n + n$ ,  $(n + n, n + 2n)$ , r or each  $n - 1, \ldots, 0$  the homogeneous  $\mathcal{O}(\epsilon)$  the implementation gives the following the following the following the following the following theorem. *<sup>i</sup> <sup>r</sup> <sup>h</sup> <sup>t</sup> <sup>k</sup> <sup>i</sup> <sup>r</sup> <sup>h</sup> <sup>t</sup> <sup>i</sup> <sup>r</sup> <sup>t</sup> <sup>i</sup> <sup>i</sup> f C x y C x y C x y C x y C x y C x y*  $1 - 2$  3  $-$  3  $P_{i,j}$  for  $r - i - 0$  equals the nomogeneous coordinate of  $e_i$  for<br> $t > 0$  $T \leq 0$ ,  $T \leq T \leq 0$ ,  $T \leq T \leq 0$ ,  $T \leq T \leq T \leq 0$ ,  $T \leq T \leq T \leq 0$ ,  $T \leq T \leq T \leq T \leq T \leq T \leq T \$  $t > 0$ . f e, for  $r = t = 0$  equals the homogeneous coordinate of e, for

**Proof.** The Newton polytope of A and the respective normal vector  $v_i$  is<br>shown in Fig. 2 where  $v_i = (1, 0)$ ,  $v_i = (0, 1)$ ,  $v_i = (1, 1)$ ,  $v_i = (1, 1)$ , and  $v_i =$ shown in Fig. 2 where  $v_1 = (1,0)$ ,  $v_2 = (0,1)$ ,  $v_3 = (-1,1)$ ,  $v_4 = (-1,-1)$ , and  $v_5 =$ (1,-1). From (6) the convex hull of A is the Newton polytope Q is defined by  $a_i = -\min_{m \in \mathcal{Q}} \langle m, v_i \rangle$  and the Q-homogenisation map  $\phi_Q : Z^2 \to Z^5$  is defined by Taking *r*,*t*= 0, the homogeneous coordinates for each element of *A*, **Proof.** The Newton polytope of *A* and the respective normal vector  $\bf{v}$  is **Proof.** The Newton polytope of A and the respective normal vector  $v_i$  is<br>shown in Fig. 2 where  $v_1 = (1,0)$ ,  $v_2 = (0,1)$ ,  $v_3 = (-1,1)$ ,  $v_4 = (-1,-1)$ , and  $v_5 =$ <br>(1,-1). From (6) the convex hull of A is the Newton polytope **Proof.** The Newton polytope of *A* and the respective normal vector *<sup>i</sup>* is shown in Fig. 2 where **Proof.** The Newton polytope of A and the respective normal vector  $v_i$  is above in Fig. 2 where  $v_i = (1, 0)$ ,  $v_i = (0, 1)$ ,  $v_i = (0, 1)$ ,  $v_i = (0, 1)$ , and  $v_i =$ *Droof* **Proof.** The Newton polytope of *A* and the respective normal vector  $v_i$  is  $(1,-1)$ . From (6) the convex hull of A is the Newton polytope *Q* is defined by  $a = -\min \langle m, v_i \rangle$  and the O-homogenisation map  $\phi_0: Z^2 \to Z^5$  is defined by  $meQ$ polytope of A and the respective normal vector  $v_i$  is  $\rho_Q: Z^2 \to Z^3$  is defined by polynomial system of the two variables of the form of the form,  $(1, 1)$ nvex null of A is the Newton polytope Q is defined by<br>he Q homogenisation man  $\phi_2$ ,  $7^2$ ,  $\sqrt{7^5}$  is defined by  $\mathcal{U}$   $\setminus$  1,2,  $\mathcal{L}$   $\math$ Newton polytope of A and the respective normal vector  $v_i$  is  $\nu_i$  and the Q-homogenisation map  $\phi_Q: Z^2 \to Z^5$  is defined by *A r*,*t*,*r h*,*t*,*r*,*t k* ,*r h*,*t k* ,*r* 2*h*,*t k* ,*r h*,*t* 2*k* . **Proof.** The Newton polytope of *A* and the respective normal vector *<sup>i</sup>* is shown in Fig. 2 where  $|v_i\rangle + a_i$  for  $i = 1,...,5$ .  $\langle v_i \rangle + a_i$  for  $i = 1,...,5$ . *For each i* 1,,6 *the homogeneous coordinate of i e for r t* 0 *equals the homogeneous*  vector  $\nu$  is resultant  $f_i =$ <br>  $\frac{1}{2}$  and  $v_i =$ Let *r*, *t* be non-negative integers and *h*, *k* be positive integers. Consider the unmixed **Theorem 4.** *Let r*,*t* 0*, where r*,*t***Z***and let h***Z***. Let* , <sup>1</sup>*<sup>f</sup>* <sup>2</sup>*fand* 3*<sup>f</sup> be an unmixed* 

 $\phi_Q(\alpha)_i = \langle \alpha, v_i \rangle + a_i$  for  $i = 1,...,5$ .  $\phi_Q(\alpha)_i = \langle \alpha, v_i \rangle + a_i$  for  $i = 1,...,5$ .  $\varphi$ <sub>*i*</sub>  $i/\tau u_i$  for  $i = 1,...,3$ . *A r*,*t*,*r h*,*t*,*r*,*t k* ,*r h*,*t k* ,*r* 2*h*,*t k* ,*r h*,*t* 2*k* .  $\sum_{i} a_i + a_i$  for  $i = 1,...,S$ . *A r*,*t*,*r h*,*t*,*r*,*t h*,*r h*,*t h*,*r* 2*h*,*t h*,*r h*,*t* 2*h*.  $P_i$ <sup>*N*</sup> *P* and the *i* is shown in Fig. 2 where  $I_i$  is shown in F  $v_i$ / +  $u_i$  1or  $i = 1,...,3$ .  $V_i$  +  $a_i$  for  $i = 1,...,5$ .

Taking  $(r,t) = 0$ , the homogeneous coordinates for each element of A, raking (*r*,*t*) – 0, the homogeneous coordinates for each element of A,<br> $\phi_Q(0,0) = (0,0,h,3h,h) = h(0,0,1,3,1)$  $\psi_Q(0,0) = (0,0,n,5n,n) = n(0,0,1,5,1)$ <br>  $\phi_Q(h,0) = (h,0,0,2h,2h) = h(1,0,0,2,2)$  $\phi_0(0,h) = (0,h,2h,2h,0) = h(0,1,2,2,0)$  $\phi_Q(h,h) = (h,h,h,h,h) = h(1,1,1,1,1)$  $\phi_Q(2h,h) = (2h,h,0,0,2h) = h(2,1,0,0,2)$  $\phi_Q(h,2h) = (h,2h,2h,0,0) = h(i,2,2,0,0)$ *<sup>Q</sup> h*,0 *h*,0,0,2*h*,2*h h*1,0,0,2,2 Taking  $(r,t) = 0$ , the homogeneous coordinates for each element of A  $\psi_0(2n,n) = (2n,n,0,0,2n)$ <br> $\phi_0(h,2h) = (h,2h,2h,0,0)$ *<sup>Q</sup>* 2*h*,*h* 2*h*,*h*,0,0,2*h h*2,1,0,0,2 *<sup>Q</sup>* 0,0 0,0,*h*,3*h*,*h h*0,0,1,3,1  $\frac{dy}{dx}$  $\phi_{\alpha}$ (*i*) *<sup>Q</sup> h*,*h h*,*h*,*h*,*h*,*h h*1,1,1,1,1 *<sup>Q</sup>* 2*h*,*h* 2*h*,*h*,0,0,2*h h*2,1,0,0,2 *<sup>Q</sup>* 0,0 0,0,*h*,3*h*,*h h*0,0,1,3,1 *<sup>Q</sup> h*,0 *h*,0,0,2*h*,2*h h*1,0,0,2,2 *<sup>Q</sup>* 0,*h* 0,*h*,2*h*,2*h*,0 *h*0,1,2,2,0 *<sup>Q</sup> h*,*h h*,*h*,*h*,*h*,*h h*1,1,1,1,1 *<sup>Q</sup>* 2*h*,*h* 2*h*,*h*,0,0,2*h h*2,1,0,0,2 is a  $(r,t) = 0$ , the homogeneous coordinates for each element of A,  $\phi_Q(h,2h) = (h,2h,2h,0,0) = h(i,2,2,0,0)$  $\phi_Q(0,h) = (0,h,2h,2h,0) = h(0,1,2,2,0)$  $\begin{aligned} \n\phi_Q(h,h) &= (h,h,h,h,h) = h(1,1,1,1,1) \\ \n\phi_Q(2h,h) &= (2h,h,0,0,2h) = h(2,1,0,0,2) \n\end{aligned}$  $(r,t) = 0$ , the homogeneous coordinates for each element of A,  $(v, t) = 0$ , the homogeneous coordinates for each element of A,<br> $\phi_Q(0,0) = (0,0,h,3h,h) = h(0,0,1,3,1)$  $\varphi_Q(2h,h) = (2h,h,0,0,2h) = h(2,1,0,0,2)$ <br>  $\varphi_Q(h,2h) = (h,2h,2h,0,0) = h(i,2,2,0,0)$  $\phi_Q(h,2h) = (h,2h,2h,0,0) = h(i,2,2,0,0)$  $\begin{aligned} \n\varphi_Q(h,0) &= (h,0,0,2h,2h) = h(1,0,0,2,2), \\ \n\varphi_Q(0,h) &= (0,h,2h,2h,0) = h(0,1,2,2,0), \\ \n\varphi_Q(h,h) &= (h,h,h,h,h) = h(1,1,1,1,1). \n\end{aligned}$  $\varphi_Q(h,2h) = (h,2h,2h,0,0) = h(i,2,2,0,0)$ <br>  $\cdot$  0. Let  $a = (r,t)$ , then the homogeneous coordinates for each  $\psi_2(h,0) = (h,0,0,2h,2h) = h(1,0,0,2,2)$ <br> $\phi_2(0,h) = (0,h,2h,2h,0) = h(0,1,2,2,0)$  $\frac{2}{2}$   $\frac{2}{2}$   $\frac{2}{2}$   $\frac{2}{2}$  $(x, \lambda) = 0$ , the homeogeneous executive to seek element of *A* **a**  $\overline{a}$  mates for each otenfient of *A*,  $\overline{a}$  *z*  $\overline{b}$  *z*  $\overline{c}$  *z*  $\overline{$  $\phi_{\mathcal{Q}}(h,0) = (h,0,0,2h,2h) = h(1,0,0,2,2)$  $(1,0, -0, 0, 1, 2L, 1) = L(0, 0, 1, 2, 1)$ <br> $\phi_{\alpha}(0,0) = (0, 0, L, 2L, 1) = L(0, 0, 1, 2, 1)$ *ai* min *m*, and the *Q*-homogenisation map <sup>2</sup> <sup>5</sup> *<sup>Q</sup>* : **Z Z** is  $\liminf$  of  $A$ , *ai* min *m*, and the *Q*-homogenisation map <sup>2</sup> <sup>5</sup> *<sup>Q</sup>* : **Z Z** is

Let  $r, t > 0$ . Let  $a = (r, t)$ , then the homogeneous coordinates for each element of 4 element of  $A$ ,  $I_{\text{at}} \sim 0$ ,  $I_{\text{at}} \sim (n\lambda)$ , then the firm **c** *r*,*t*<sub>1</sub>,  $\phi_0(a) = \angle (a \pm b) (1 \pm 0) > 1 \pm (1 \pm 0) = 0$  $\ge 0$ . Let  $a = (r,t)$ , then the homogeneous coordinates for each  $\int f \, d\mathbf{f}$ , then the homogeneous coordinates for  $A$ , 1,0, <sup>1</sup> 0,1, <sup>2</sup> 1,1, <sup>3</sup> 1, 1 <sup>4</sup> and 1, 1 <sup>5</sup> . From (6) the convex hull of *A* is the **Proof.** The Newton polytope of *A* and the respective normal vector *<sup>i</sup>* is shown in Fig. 2 where *ai* min *m*, and the *Q*-homogenisation map <sup>2</sup> <sup>5</sup> *<sup>Q</sup>* : **Z Z** is *For each i* 1,,6 *the homogeneous coordinate of i efor r t* 0 *equals the homogeneous*   $\alpha$   $\alpha$   $\alpha$   $\alpha$   $\beta$   $\gamma$ ,  $\beta$  and the nomogeneous coordinates for each vector  $\alpha$  $\phi_{\alpha}(x) = \angle (x)$  (1.0)  $\times$   $\pm (x) = 0$ *<sup>Q</sup>* 0,*h* 0,*h*,2*h*,2*h*,0 *h*0,1,2,2,0 *<sup>Q</sup> h*,*h h*,*h*,*h*,*h*,*h h*1,1,1,1,1 *<sup>Q</sup>* 2*h*,*h* 2*h*,*h*,0,0,2*h h*2,1,0,0,2  $\phi$ , then the homogeneous coordin *<sup>Q</sup> h*,0 *h*,0,0,2*h*,2*h h*1,0,0,2,2  $a = (r,t)$ , then the homogeneous coordinates for each *g*  $(1,0)$   $\leq$   $(1,0)$   $\leq$   $(0,1)$  $\text{It is for each }$ 

$$
\phi_Q(a)_1 = \langle (r,t), (1,0) \rangle + (-r) = 0
$$
  
\n
$$
\phi_Q(a)_2 = \langle (r,t), (0,1) \rangle + (-r) = 0
$$
  
\n
$$
\phi_Q(a)_3 = \langle (r,t), (-1,1) \rangle + (r-t+h) = h
$$
  
\n
$$
\phi_Q(a)_4 = \langle (r,t), (-1,-1) \rangle + (r+t+3h) = 3h
$$
  
\n
$$
\phi_Q(a)_5 = \langle (r,t), (1,-1) \rangle + (-r+t+h) + h
$$
  
\n
$$
\therefore \phi_Q(r,t) = (0,0,h,3h,h).
$$

Continuing the same computation for other exponent vectors the following homogeneous coordinates are derived for each a,  $\phi_Q$  ( $r + h, t$ )  $\rho = (0,0,h,3h,h)$ .<br>e same computation for other exponent vectors the  $\alpha$  *dependence and depited for each a* because coordinates are derived for each a,  $\phi_Q$   $(r + h,t)$ 

 $=(h,0,0,2h,2h), \phi_{\mathcal{Q}}(r,t+h)=(0,h,2h,2h,0), \phi_{\mathcal{Q}}(r+h,t+h)=(h,h,h,h,h),$  $\phi_Q$   $(r + 2h, t + h) = (2h, h, 0, 0, 2h)$  and  $\phi_Q$   $(r + h, t + 2h) = (h, 2h, 2h, 0, 0, 0, 0, 0, 0, 0)$  $\mathcal{L}$  3  $\mathcal{L} \left( \mathcal{L} \right)$ generalised. The main theorem that established the conditions that can give a determinantal hybrid  $h(0,0,2h,2h)$ ,  $\phi_Q(r,t+h)=(0,h,2h,2h,0)$ ,  $\phi_Q(r+h,t+h)=(h,h,h,h,h)$ ,  $T_{\rm F}$  results on the resultant matrix are  $T_{\rm F}$  and  $T_{\rm F}$  are  $T_{\rm F}$  and  $T_{\rm F}$  are  $T_{\rm F}$  are  $T_{\rm F}$  $\phi_Q$   $(r + 2h, t + h) = (2h, h, 0, 0, 2h)$  and  $\phi_Q$   $(r + h, t + 2h) = (h, 2h, 2h, 0, 0)$  $g = (0, n, 2n, 2n, 0), \varphi_Q(r + n, t + n) = (n, n, n, n, n),$  $(0, 0, 2h, 2h)$   $\phi_0(r + h) = (0, h, 2h, 0)$   $\phi_0(r + h + h) = (h, h, h, h)$ Thus,  $e^{\phi(t)} = (h,0,0,2h,2h)$  ,  $\phi_{Q}(r,t+h) = (0,h,2h,2h,0)$ ,  $\phi_{Q}(r+h,t)$ 

> Thus, Thus, Taking *r*,*t*= 0, the homogeneous coordinates for each element of *A*, Thus,  $\frac{1}{2}$ ,  $\frac{1}{2$  $Thus,$ Let *r*, *t* be non-negative integers and *h*, *k* be positive integers. Consider the unmixed

$$
\begin{aligned}\n\phi_Q(0,0) &= \phi_Q(r,t) = (0,0,1,3,1) \\
\phi_Q(h,0) &= \phi_Q(r+h,t) = (1,0,0,2,2) \\
\phi_Q(0,h) &= \phi_Q(r,t+h) = (0,1,2,2,0) \\
\phi_Q(h,h) &= \phi_Q(r+h,t+h) = (1,1,1,1,1) \\
\phi_Q(2h,h) &= \phi_Q(r+2h,t+h) = (2,1,0,0,2) \\
\phi_Q(h,2h) &= \phi_Q(r+h,t+2h) = (1,2,2,0,0).\n\end{aligned}
$$

**Theorem 5.** Let  $f_1, f_2$  and  $f_3$  be an unmixed polynomial system with support  $A = \{(0,0), (h,0), (0,h), (h,h), (2h,h), (h,2h)\}$ . For each  $i = 1,...,6$ support  $A = \{(0,0), (n,0), (0,n), (n,n), (2n,n), (n,2n)\}$ . For each  $1 = 1,...,6$ <br>the homogeneous coordinate of  $e_1$  for  $h=1$ , equals the homogeneous coordinate of  $e_1$  for any  $h > 1$ . Let *r*,*<sup>t</sup>* 0. Let *r*,*t*, then the homogeneous coordinates for each element of *A*, 10  $\geq l$ ,  $\geq$  1,  $\geq$   $\alpha h > 1$ . and  $J_3$  be an unmixed polynomial system with  $\frac{1}{2}$  of the implementation gives the following theorem.  $\sim I$ upport  $A = \{(0,0), (0,0), (0,0), (1,0), (2,0,1), (1,2,0)\}$ . For each  $t = 1,...,0$ <br>he homogeneous coordinate of e for  $h=1$  equals the homogeneous  $\sim$ **eorem 5**. Let  $f_1, f_2$  and  $f_3$  be an unmixed polynomial system with geneous coordinate of  $e_1$  for  $n-1$ , equals the nomogeneous<br>te of e. for any  $h > 1$ .  $A = \{(0,0), (h,0), (0,h), (h,h), (2h,h), (h,2h)\}\$ . For each  $i = 1,...,6$ **Proof.** For *h* 1 , name the support of the system *A*1given by *A* 0,0,1,0,0,1,1,1,2,1,1,2.  $T$  **1.**  $\mathbf{f} \times \mathbf{f} \times \mathbf{f} \times \mathbf{f}$  and  $\mathbf{f} \times \mathbf{f} \times \mathbf{f}$ **Theorem 5.** Let  $f_1, f_2$  and  $f_3$  be an unmixed polynomial system with<br>support  $A = f(0,0)$  (b, 0) (0, b) (b, b) (b, b) (b, b)? For each  $i = 1$  6 the homogeneous coordinate of  $e_i$ , for  $h = 1$ , equals the homogeneous

**Proof.** For  $h = 1$ , name the support of the system  $A_1$  given by  $A = \{(0, 1)\}$ **Proof.** For  $n = 1$ , hame the support of the system  $A_1$  given by  $A = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 1), (1, 2)\}$ . Homogeneous coordinate for each element of  $A_1$ ,  $\phi_Q(0,0) = (0,0,1,3,1), \phi_Q(1,0) = (1,0,0,2,2), \phi_Q(0,1) = (0,1,2,2,0),$  $\phi_Q$   $(1,1) = (1,1,1,1,1), \phi_Q(2,1) = (2,1,0,0,2), \phi_Q(1,2) = (1,2,2,0,0).$ **Proof.** For  $h = 1$ , name the support of the system  $A_1$  given by  $A = \{(0,0), (0,1), (1,1), (2,1), (1,2)\}$ . Homogeneous coordinate for each element *ai* min *m*, and the *Q*-homogenisation map <sup>2</sup> <sup>5</sup> *<sup>Q</sup>* : **Z Z** is **Proof.** For  $h = 1$ , name the support of the system  $A_1$  given by  $A = \{(0, 0, 1, 0), (0, 1), (1, 1), (2, 1), (1, 2)\}$ . However, and the formal proof above the system of  $\varphi_Q$  (1,1) = (1,1,1,1,1),  $\varphi_Q$ (2,1) = (2,1,0,0,2),  $\varphi_Q$  (1,2) = (1,2,2,0,0). *ai* min *m*, and the *Q*-homogenisation map <sup>2</sup> <sup>5</sup> *<sup>Q</sup>* : **Z Z** is Newton polytope *<sup>Q</sup>* is defined by *<sup>i</sup> <sup>m</sup> <sup>Q</sup>* Taking *h* > 1, the homogeneous coordinates of each element of *Ah* are,

Taking  $h > 1$ , the homogeneous coordinates of each element of  $A<sub>h</sub>$ are  $\phi_Q(h,0) = h(1,0,0,2,2), \phi_Q(h,0) = h(0,1,2,2,0), \phi_Q(h,h) = h(1,1,1,1,1),$  $h(2h,h) = h(2,1,0,0,2), \ \phi_Q(h,2h) = h(1,2,2,0,0).$ Taking  $h > 1$ , the homogeneous coordinates of each element of  $A_h$  $\phi_2(2,2)$ ,  $\phi_2(0,1)$   $h(0,1,2,2,0)$ ,  $h(0,2)$ <br> $h(0,2)$   $h(0,2)$   $h(1,2,2,0,0)$ *<sup>Q</sup>* 0,*h* 0,*h*,2*h*,2*h*,0 *h*0,1,2,2,0 *<sup>Q</sup>* 0,*h* 0,*h*,2*h*,2*h*,0 *h*0,1,2,2,0  $\phi_Q$  (2*h*,*h*) = *h*(2,1,0,0,2),  $\phi_Q$ (*h*,2*h*) = *h*(1,2,2,0,0).  $\alpha$  minute  $\beta$   $\beta$ *<sup>Q</sup> h*,0 *h*1,0,0,2,2, *<sup>Q</sup>* 0,*h h*0,1,2,2,0, *<sup>Q</sup> h*,*h h*1,1,1,1,1 , *<sup>Q</sup>* 2*h*,*h h*2,1,0,0,2,

Theorem 4 illustrates that scaling the edges of  $A_1h$  times gives the same homogeneous coordinate since for all  $\hat{e}_i \in A_{\hat{h}}$ ,  $\phi(\hat{e}_i) = h\phi(e_i) = \phi(e_i)$  by nomogeneous coordinate since for all  $e_i \in A_h$ ,  $\varphi(e_i) = n\varphi(e_i) = \varphi(e_i)$  by  $\varphi$  . Definition 1. Each homogeneous coordinate is a point in  $\mathbf{P}^4$  which is written Definition 1. Each homogeneous coordinate is a point in T which is written in the following nonlinear uniquely up to multiplication by  $\lambda \in \mathbb{C}^*$ . Here  $\lambda = h$ .  $\mu$  *h*,  $\mu$ <sub>1</sub>,  $\mu$ <sub>2</sub>,  $\mu$ <sup>1</sup> <sup>2</sup> *f b b x b y b x y b x y b x y <sup>i</sup> <sup>i</sup> <sup>i</sup> <sup>i</sup> <sup>i</sup> <sup>i</sup>* , *i* 1,2,3. ch homogeneous coordinate is a point in  $\mathbf{P}^4$  which is written<br>multiplication by  $2 \in \mathbb{C}^*$ . Hore  $2 = h$  $\alpha$  *h*, 2*h*, 2  $\lambda$ , 11ClC  $\lambda - h$ , denote that scaling the edges of  $A$ .  $\Phi$  *h*  $\Phi$  *h*  $\Phi$  *c h*  $\Phi$  *i h*<sub>2</sub>*h*, *y*  $\Phi$  *l*<sub>2</sub>*h*, *y h*)*n h*<sup>2</sup> *h*<sub>2</sub>*h*, *n*<sup>2</sup> *h*<sub>2</sub>  $\overrightarrow{P}$ ,  $\overrightarrow{P}$ , geneous coordinate since for all  $\hat{e}_i \in A_{\hat{h}}$ ,  $\phi(\hat{e}_i) = h\phi(e_i) = \phi(e_i)$  by  $\text{Here } \lambda = h$  $\Phi$  *documate since for an*  $\epsilon_i \in A_h$ *,*  $\psi(\epsilon_i) = h\psi(\epsilon_i) = \psi(\epsilon_i)$  *by*  $\Phi$  *and*  $\phi$  *<i>d*  $\phi$  *h*) *m h* which is written intion 1. Each homogeneous coordinate is a point in  $P^+$  which is written<br>rely un to multiplication by  $\lambda \in C^*$ . Here  $\lambda = h$ uniquely up to multiplication by  $\lambda \in \mathbb{C}^*$ . Here  $\lambda = h$ . **Example 2**. Consider the following nonlinear unmixed system, Theorem 4 illustrates that scaling the edges of  $A_1 h$  times the same homogeneous the same homo coordinate since for all  $e_i \nightharpoonup a_h$ ,  $\psi(e_i) - r$ <br>Definition 1. Each homogeneous coordinate is a noint in F

**Example 2.** Consider the following nonlinear unmixed system, Let *r*, constant the forthwing nontinear annumber system, Let *r*,*t* 0. Let *r*,*t*, then the homogeneous coordinates for each element of *A*, Let *r*,*<sup>t</sup>* 0. Let *r*,*t*, then the homogeneous coordinates for each element of *A*, 10 Let *r*,*t* 0. Let *r*,*t*, then the homogeneous coordinates for each element of *A*, nsider the following nonlinear unmixed system,  $\mathbf{\ddot{y}}$  $\ddot{\phantom{a}}$  $\mathbf n$ *b llowing nonlinear unmixed system,*  $\overline{a}$ .<br>n  $\mathbf{u}$ Example 2. Consider the following nonlinear unmixed system,

$$
f_i = b_{i1} + b_{i2}x^2 + b_{i3}y^2 + b_{i4}x^2y^2 + b_{15}x^2y^2, i = 1,2,3.
$$

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This system is generated when  $h = 2$  ( $h > 1$ ) and gives support  $A =$  ${(0,0), (2,0), (0,2), (2,2), (4,2), (2,4)}$ . By Definition 2, the homogeneous coordinates for each support are shown in Table 3.

Exponent vector	Homogeneous coordinate
a	$(a_1,,a_s)$
(0, 0)	$(0, 0, 2, 6, 2) = 2(0, 0, 1, 3, 1)$
(2, 0)	$(2, 0, 0, 4, 4) = 2(1, 0, 0, 2, 2)$
(0, 2)	$(0, 2, 4, 4, 0) = 2(0, 1, 2, 2, 0)$
(2, 2)	$(2, 2, 2, 2, 2) = 2(1, 1, 1, 1, 1)$
(4, 2)	$(4, 2, 0, 0, 4) = 2(2, 1, 0, 0, 2)$
(2, 4)	$(2, 4, 4, 0, 0) = 2(1, 2, 2, 0, 0)$

**Table 3: Homogeneous Coordinates**

This example showed that by scaling the edges,  $h = 2$  times gives the same homogeneous coordinate for  $h = 1$ .

# **Conclusion**

The concepts in algebraic geometry were highlighted through the construction of Bèzout matrix. The implementation of the algorithm of Bèzout matrix is on the unmixed bivariate polynomial systems of the form equation (8) by considering the origin as the distinguished point of the Newton polytope. Detail implementation and construction are presented to indicate that, it is sufficient to consider only the case when  $r = t = 0$  and  $h = k$  and scaling the edges of  $A_1$ , h times gives the same homogeneous coordinate.

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